Discrete Mathematics

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Relations

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Contents

- Binary relations
- Relations on a set
- Properties of relations
- Composite relations
- n-ary relations and operations
- Representing relations
- Closures of relations
- Equivalence relations, classes, partitions
- Spatial relations

Relationships

- Relationships between elements of sets occur in many contexts:
  - the set of employees and the set of their salaries
  - flight numbers and take off times
  - students and subjects
  - a real number and integers that are larger than it
  - a computer language and a valid statement in this language

- Relationships between elements of sets are represented using the discrete structure called a relation

Cartesian Product of Sets

For sets *A*, *B*, their *Cartesian product*

$A \times B :\equiv \{(a, b) \mid a \in A \wedge b \in B\}$

is the set of all possible ordered pairs whose first component is a member of *A* and whose second component is a member of *B*

Example:

Flights = {KL108, BA9, AF11}

Time = {12, 13}

Flights×Time = { (KL108,12),(KL108,13), (BA9,12),(BA9,13), (AF11,12),(AF11,13) }

Binary Relations

- Let *A*, *B* be any sets. *Binary relation R* from *A* to *B*, *R*:*A*×*B* can be identified with a subset of *A*×*B:*

$$R \subseteq \{(a,b) \mid a \in A \wedge b \in B\}$$

- (*a*,*b*)∈*R* means that *a* is related to *b* (by *R*)
- Relation also written as *aRb* or *R(a,b):*

$$a\,R\,b \Leftrightarrow (a,b) \in R$$

- Example:
  - *relation* **<** can be seen as {(a,b) | a < b}
  - a**<**b and **< (**a,b) both mean (a,b)∈ **<**

Defining a binary relation:

- Make a list of all pairs (a,b)
- If P(x1, x2) is a predicate with two variables, then

  {(a, b) | P(a,b)} is a binary relation
- A binary relation *R* corresponds to a characteristic function $P_R: A \times B \rightarrow \{\mathbf{T},\mathbf{F}\}$ defining a subset *R* with True values

Example 1

Flights = {KL108, BA9, AF11}

Time = {12, 13}

Cartesian product

Flights$\times$Time =

{ (KL108,12),(KL108,13),(BA9,12),(BA9,13),(AF11,12),(AF11,13) }

Relation

Takeoff (Flights, Time) $\subseteq$ Flights $\times$ Time

Takeoff (Flights, Time) =

{ (KL108,12), (BA9,13), (AF11,13) }

Example 2

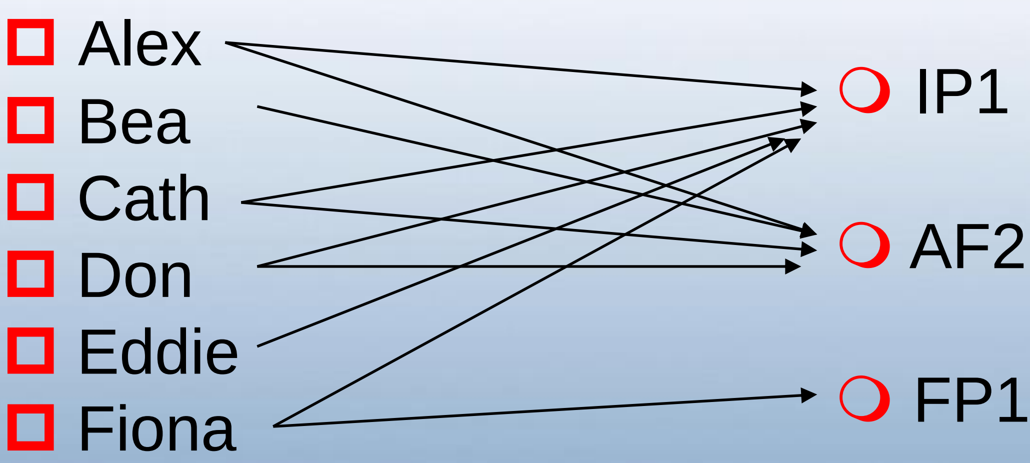
- Students A:
  - Alex, Bea, Cath, Don, Eddie, Fiona
- Subjects B:
  - IP1, FP1, AF2
- Let R be the relation of students who passed subjects

R = {(Alex,IP1),(Alex,AF2),(Bea,AF2),(Cath,AF2),(Cath,IP1),
(Don,AF2),(Don,IP1),(Fiona,IP1),(Eddie,IP1),(Fiona,FP1)}

- |AxB| = 18, |R| = ?
- Order between pairs is insignificant (look at Fiona)
- Order within pairs ***is*** significant (a pair (FP2,Fiona)?)

Binary Relations

R = {(Alex,IP1),(Alex,AF2),(Bea,AF2),(Cath,AF2),(Cath,IP1),
(Don,AF2),(Don,IP1),(Fiona,IP1),(Eddie,IP1),(Fiona,FP1)}



It is not a function

But you could have functional relations

Relations on a Set

- A (binary) relation from a set *A* to itself is called a relation *on* the set *A*.
- Binary relation *R* on set *A*, $R{:}A\times A$ is a subset of $A\times A$:

$$R \subseteq \{(a,b) \mid a\in A \wedge b\in A\}$$

- Example: the "**<**" relation can be defined as a relation *on* the set **N** of natural numbers.

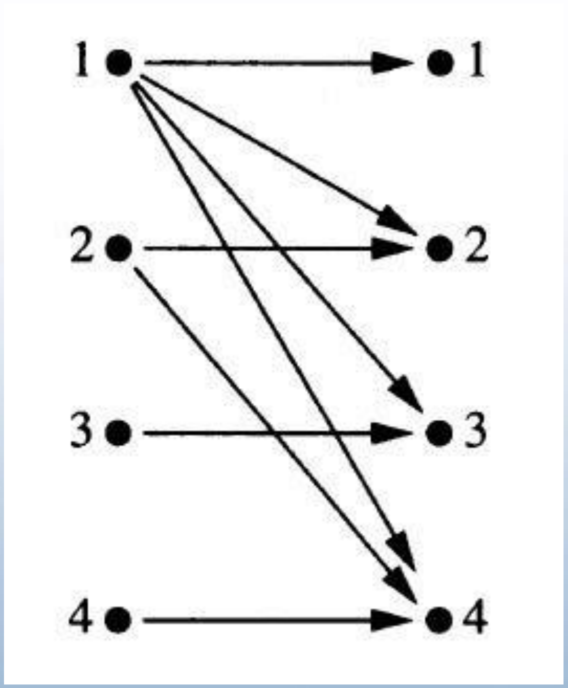
Example:

Let A be the set { 1, 2, 3, 4}.

Which ordered pairs are in the relation on the set A:

R = {(a , b) I a divides b} ?

R = { (1,1), (1,2), (1,3), (1,4), (2, 2), (2,4), (3 ,3), (4, 4) }



Number of Relations on a Set

How many relations are there on a set of n elements?

When we have a relation on a single set A

- each relation R_i is a subset of $\{(x,y) | x \in A \wedge y \in A\}$
- the cardinality of A × A is $|A \times A| = n^2$
- there are 2^k subsets of a set of size k
- if the set of tuples to choose from is of size $k=n^2$
- then there are 2^{n^2} possible subsets

There are 2^{n^2} possible relations on a set of n elements

$$2^{n^2}$$

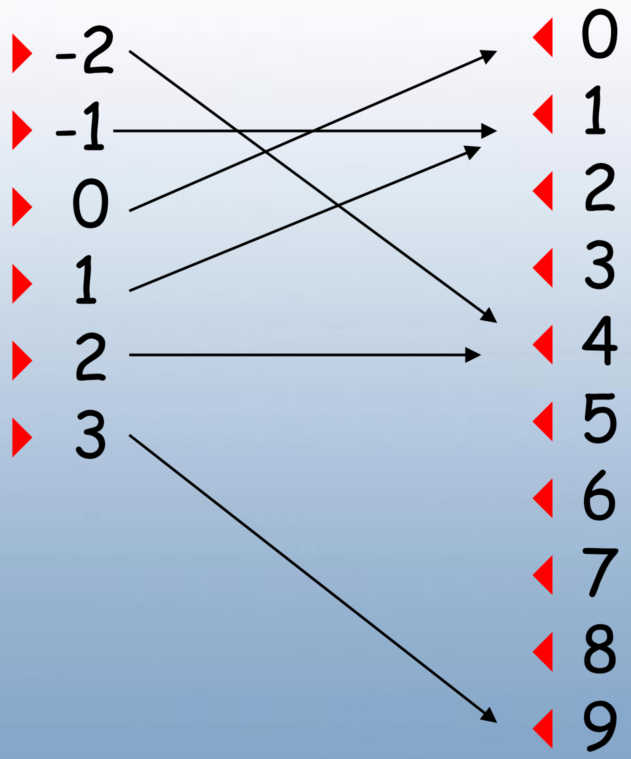
Functions as Relations

- Recall: function from a set *A* to a set *B* assigns exactly one element of *B* to each element of *A*
- We can represent a function explicitly by listing for each value *a* in the domain *A* its image *b* in the co-domain *B*
- That is we can represent the function as a set of pairs *(a,b)* as a binary relation

Functions as Relations

F: $A \rightarrow B$

- where $f(x) = x^2$
- A = {-2,-1,0,1,2,3} and B = {0,1,2,3,4,5,6,7,8,9}



R = {(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)}

Representing Relations

- General ways to represent *n*-ary relations:
  - With a list of n-tuples.
  - With a function from the (n-ary) domain to {**T**,**F**}.
- Special ways to represent binary relations:
  - With a zero-one matrix.
  - With a directed graph.

Representing Relations

- Why bother with alternative representations? Is one not enough?
- One reason: calculations are easier using one representation, other things are easier using another representation
- Matrices are appropriate for the representation of relations in computer programs
- Directed graphs are useful for understanding the properties of these relations.

Zero-One Matrices

- To represent a binary relation $R{:}A\times B$ by a 0-1 matrix $M_R = [m_{ij}]$ of size $|A|\times|B|$, let $m_{ij} = 1$ if $(a_i,b_j)\in R$ and $m_{ij} = 0$ otherwise.
- Example:
  Joe likes Susan and Mary, Fred likes Mary and Sally.
- Then the 2×3 matrix representation of the relation Likes:Boys×Girls is:

$$\begin{array}{c} \\ \text{Joe} \\ \text{Fred} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Susan} & \text{Mary} & \text{Sally} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Using Zero-One Matrices

- Special case: 0-1 matrices for relations $R{:}A\times A$
- *Convention*: rows and columns list elements of *A* in the same order
- Square matrix $n\times n$, where $n = |A|$

Example:

$$A = \{a, b, c\}$$

$$R = \{(a,a),(b,c),(c,c)\}$$

R	a	b	c
a	1	0	0
b	0	0	1
c	0	0	1

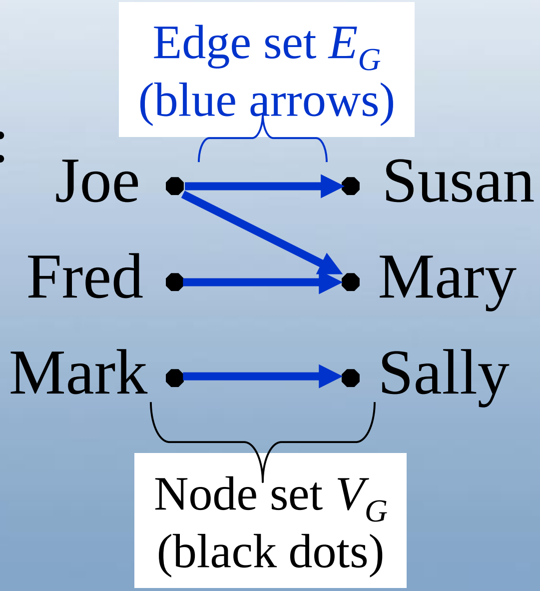
Directed Graphs

- *Directed graph* or *digraph* $G=(V_G,E_G)$ is a set V_G of *vertices (nodes)* with a set $E_G \subseteq V_G \times V_G$ of *edges (links, arcs)*. Visually represented using dots for nodes, and arrows for edges. A relation $R:A \times B$ can be represented as a graph $G_R=(V_G=A \cup B,\ E_G=R)$.

Matrix representation M_R:

	Susan	Mary	Sally
Joe	1	1	0
Fred	0	1	0
Mark	0	0	1

Graph rep. G_R:



Representing Relations

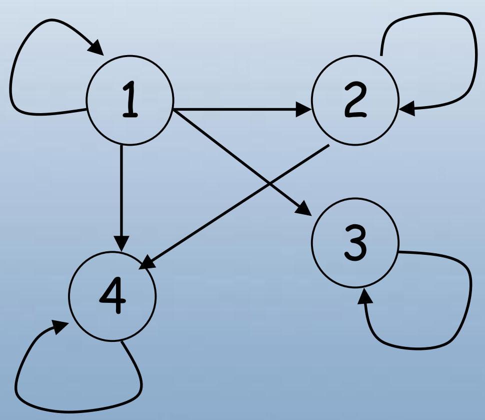
Example for R:$A \times A$:

$A = \{1,2,3,4\}$

R is the relation “a divides b” on set A:

$R = \{(a,b) \mid a \text{ divides } b,\ a \in A \wedge b \in A\}$

R = {(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)}



R	1	2	3	4
1	1	1	1	1
2	0	1	0	1
3	0	0	1	0
4	0	0	0	1

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Spatial Relations

- *Spatial relation* $R{:}G \times G$ can be identified as a subset of the Cartesian product $G \times G$:

$$R \subseteq \{(g1, g2) \mid g1 \in G \wedge g2 \in G\}$$

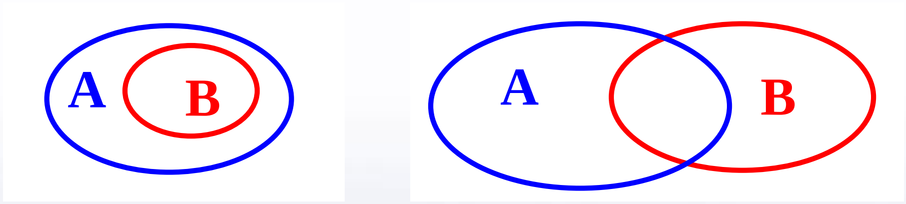
where G is a set of all subsets (geometric objects) of some geometric space (for example 3D Euclidean space)

- A binary relation R corresponds to a characteristic function $P_R{:}G \times G \rightarrow \{\mathbf{T}, \mathbf{F}\}$ defining a subset *of* all pairs *(g1,g2)* with True values

Example: relation *L1||L2* includes all pairs of parallel straight lines in the Euclidean space

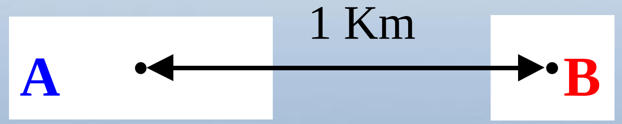
Spatial Relations

- <u>Topological Relations</u>: containment, overlapping, etc.

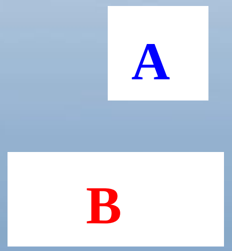


- <u>Geometric Relations</u>: parallelism ||, orthogonality ⊥

- <u>Metric Relations</u>: distance between objects, etc.

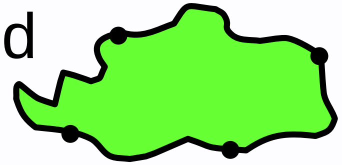


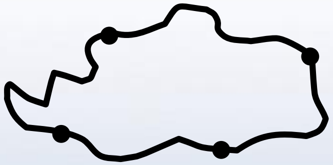
- <u>Direction Relations</u>: north of, south of, etc.



Topological Relations

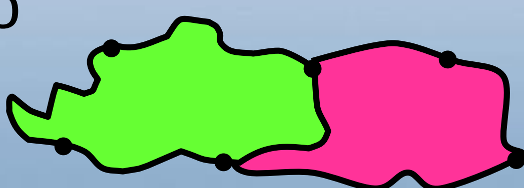
- Topological relations are defined using *point-set topology* concepts, such as *boundary*, *interior* and *exterior:*
  - *Boundary* of a region consists of a set of points that separate the region from the rest of the space





  - *Interior* of a region consists of all points in the region that are not on its boundary
  - *Exterior* of a region consists of all points that are neither boundary nor interior

- Example: *adjacency* relation means that two regions share a part of a boundary but do not share any points in their interior, also called *meet* or *touch* relation



Point Inclusion

- *Point inclusion relation*

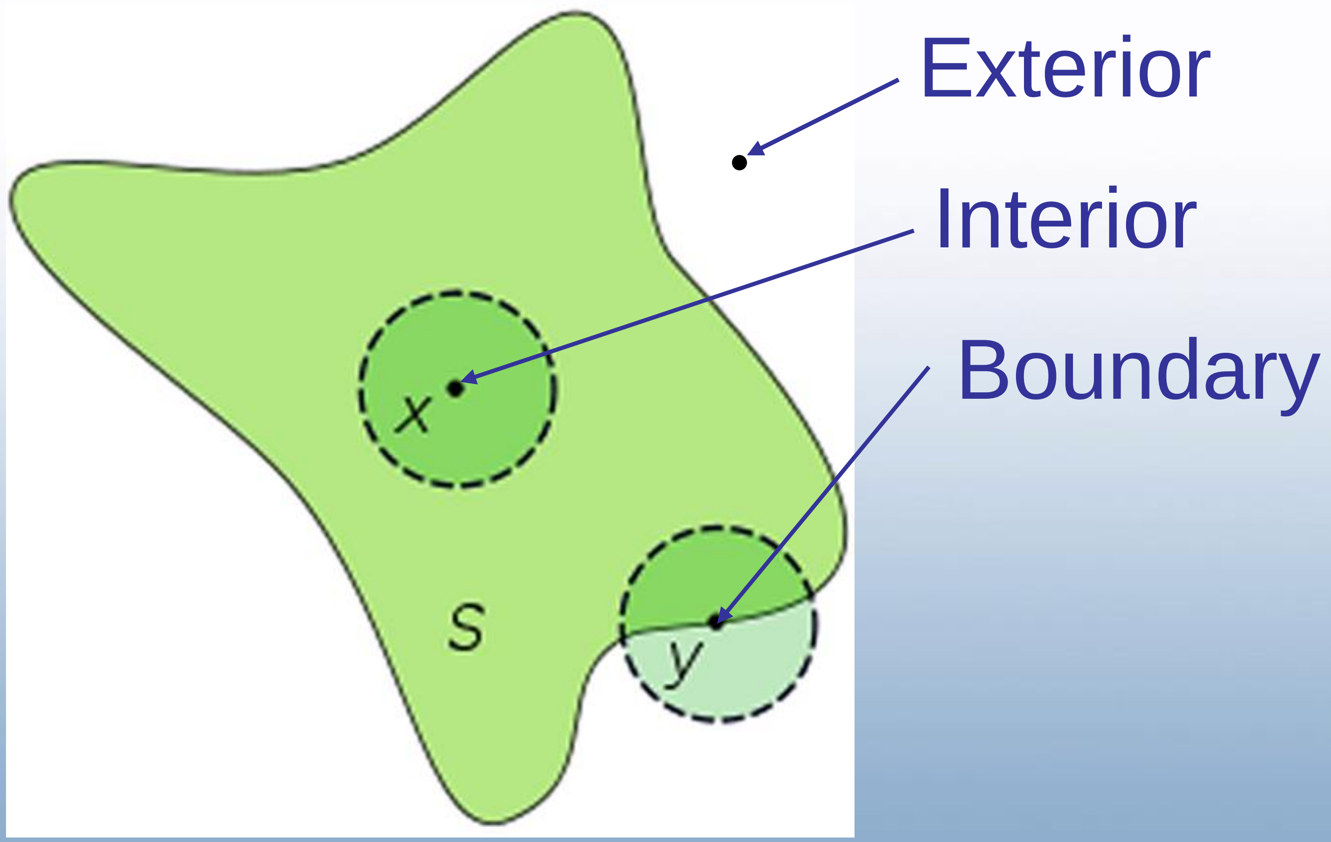
$$R \subseteq \{(p,g) \mid p \in P \wedge g \in G \wedge p \in g\}$$

where *P* is a set of all points of some geometric space, *G* is a set of subsets (geometric objects) of the geometric space, *p* is a point and *g* is a geometric object.

- Defined by a binary predicate:

$$S_2(p,g) = \begin{cases} 1, p \in g \\ 0, p \notin g \end{cases}$$

Exterior, Interior, Boundary



Neighbourhood is an open ball centered at a point

Exterior, Interior, Boundary

- *Exterior* *e(g)* of the point set *g* is a set of points, which are not contained within *g* (exterior points)
- *Interior* *i(g)* of the point set *g* is a set of points *p*, where *p* is contained within *g* together with some neighbourhood (interior points)
- *Boundary* *b(g)* of the point set *g* is a set of points *p*, where any neighbourhood of *p* contains both exterior and interior points (boundary points)
- Example: for lines, the boundary of a line consists of its endpoints, the interior of a line consists of all points composing the line excluding its endpoints.

Point Membership

- *Point membership* property is defined by a three-valued predicate:

$$S_3(p,g) = \begin{cases} 0, p \notin g, p \in e(g) \\ 1, p \in b(g) \\ 2, p \in i(g) \end{cases}$$

- Related to *ternary logic* operating with 3 values (True, False, Unknown)
- Can be considered as defined by two

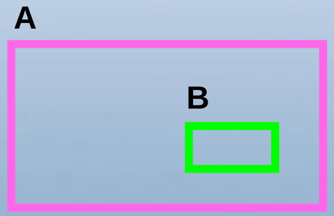
binary predicates: $S_2(p,g)$ and $S_2(p,i(g))$

Basic classification

1. DISJOINT: boundaries and interiors do not intersect

Disjoint

2. CONTAINS: interior and boundary of one object is completely contained in the interior of other object



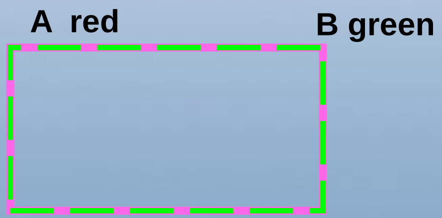
A Contains B
B Inside A

Basic classification

3. INSIDE: opposite of CONTAINS; A INSIDE B implies B CONTAINS A

A Inside B
B Contains A

4. EQUAL: the two objects have the same boundary and interior

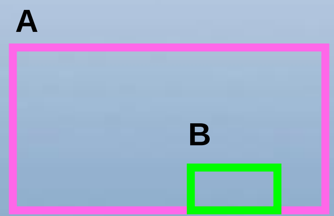


Equal

5. MEET: boundaries intersect, but interiors do not intersect

Meet (Touch)

6. COVERS: interior of one object is completely contained in interior of other object and their boundaries intersect

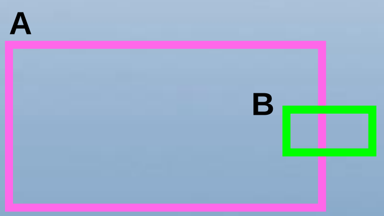


A Covers B
B Covered by A

7. COVERED BY: opposite of COVERS; A COVERED BY B implies B COVERS A

A Covered by B
B Covers A

8. OVERLAP: boundaries and interiors of the two objects intersect



Overlap

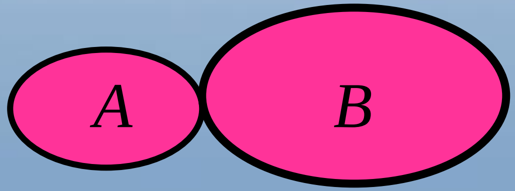
4-Intersection Matrix

- 4-intersection matrix for topological relations between point sets
- Defined on the basis of intersections between boundary and interior of two point sets *A* and *B* involved

$$\begin{pmatrix} b(A) \cap b(B) & b(A) \cap i(B) \\ i(A) \cap b(B) & i(A) \cap i(B) \end{pmatrix}$$

Each entry in the matrix is either empty or non-empty

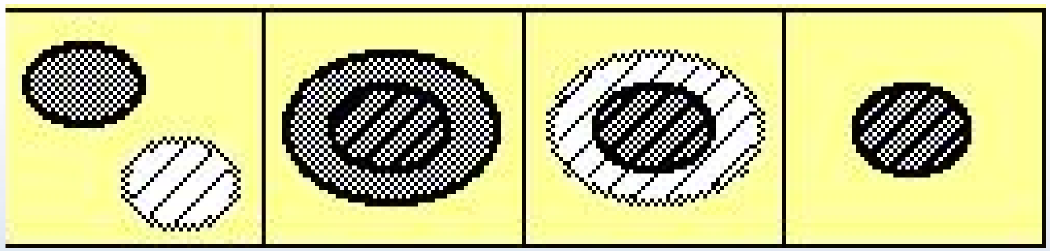
Example: $\begin{pmatrix} \neg\varnothing & \varnothing \\ \varnothing & \varnothing \end{pmatrix}$



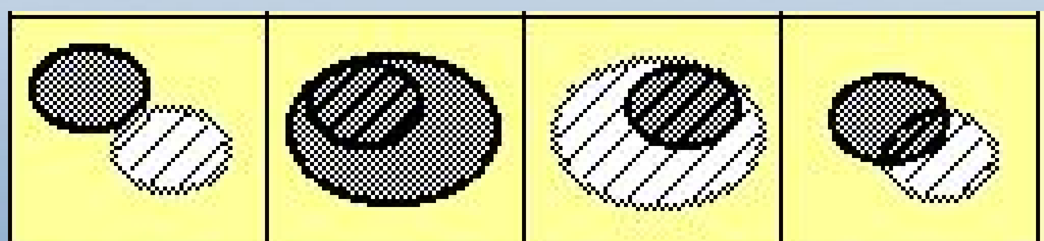
Meet(A,B)

4-Intersection Matrix

- 16 (2^4) possible matrix configurations, but only 8 are possible for <u>regions without holes</u>



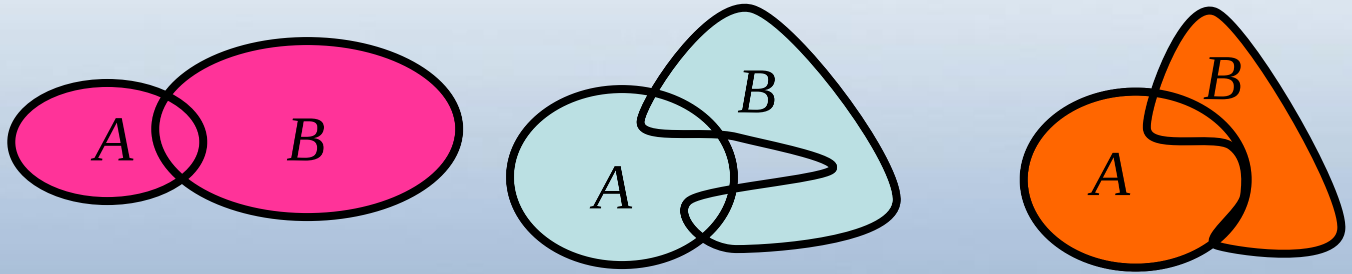
Disjoint $\begin{pmatrix} \emptyset & \emptyset \\ \emptyset & \emptyset \end{pmatrix}$ ***Contains*** $\begin{pmatrix} \emptyset & \emptyset \\ \neg\emptyset & \neg\emptyset \end{pmatrix}$ ***Inside*** $\begin{pmatrix} \emptyset & \neg\emptyset \\ \emptyset & \neg\emptyset \end{pmatrix}$ ***Equal*** $\begin{pmatrix} \neg\emptyset & \emptyset \\ \emptyset & \neg\emptyset \end{pmatrix}$



Meet $\begin{pmatrix} \neg\emptyset & \emptyset \\ \emptyset & \emptyset \end{pmatrix}$ ***Covers*** $\begin{pmatrix} \neg\emptyset & \emptyset \\ \neg\emptyset & \neg\emptyset \end{pmatrix}$ ***Covered by*** $\begin{pmatrix} \neg\emptyset & \neg\emptyset \\ \emptyset & \neg\emptyset \end{pmatrix}$ ***Overlap*** $\begin{pmatrix} \neg\emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset \end{pmatrix}$

4-Intersection Matrix

- Pros:
  - simple model
  - well accepted
- Cons:
  - Does not distinguish between conceptually different situations:

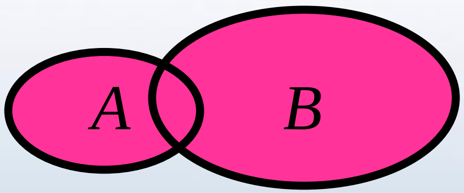


- All three situations correspond to the same matrix:

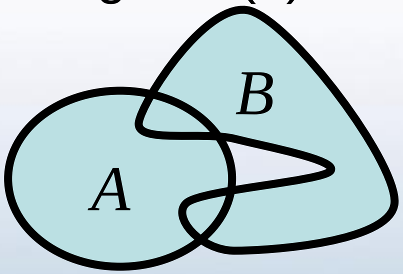
$$\begin{pmatrix} \neg\emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset \end{pmatrix}$$

Use different values for matrix entries:

- for example, number of connected components of the intersections can be used to distinguish (1) and (2)

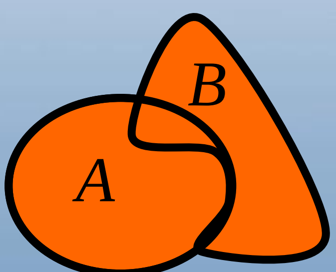


(1)



(2)

- adding the dimension of each component would distinguish from case (3)



(3)

Note: boundary intersection dimension 1 (curve)

9-Intersection Matrix

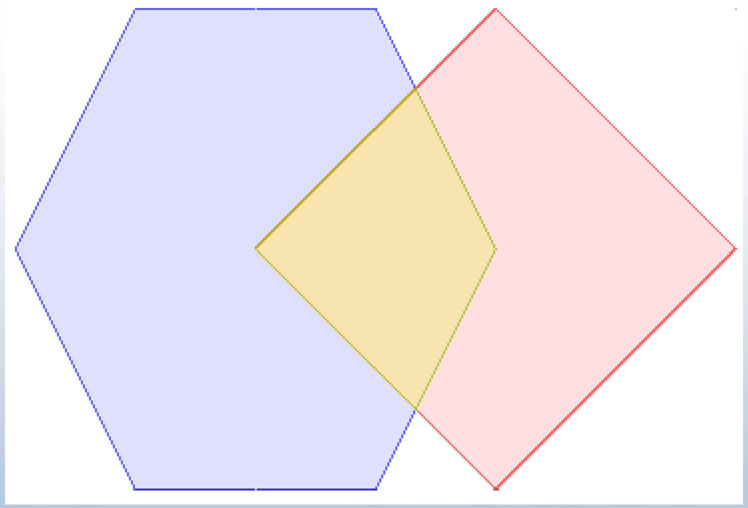
- 9-intersection matrix for topological relations between point sets and considers interior, boundary, exterior

$$\begin{pmatrix} i(A) \cap i(B) & i(A) \cap b(B) & i(A) \cap e(B) \\ b(A) \cap i(B) & b(A) \cap b(B) & b(A) \cap e(B) \\ e(A) \cap i(B) & e(A) \cap b(B) & e(A) \cap e(B) \end{pmatrix}$$

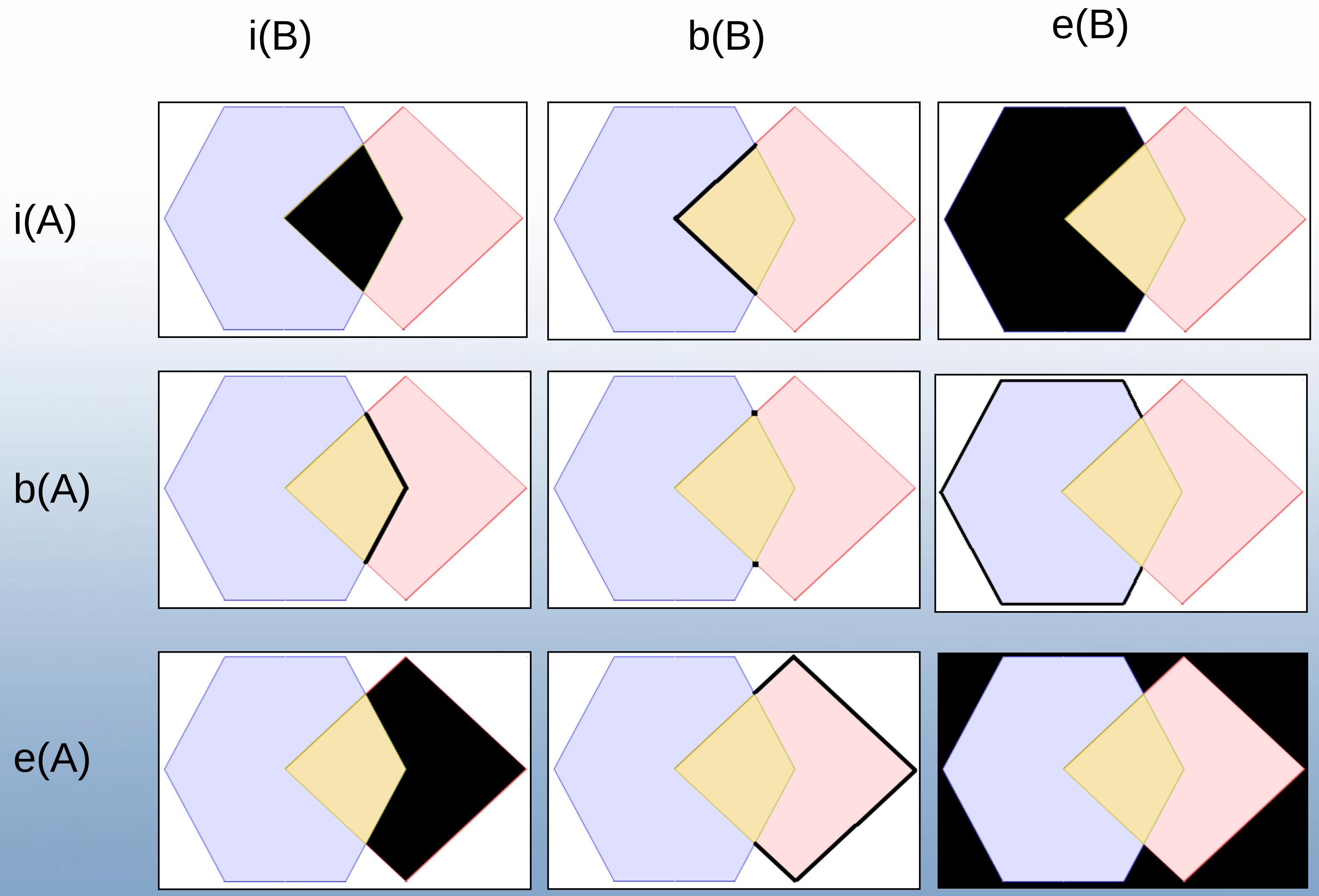
- Entries in the matrix can assume values empty/non-empty or correspond to other properties such as number of components and dimension of each component

Example

- Consider two polygons
  - A - POLYGON ((10 10, 15 0, 25 0, 30 10, 25 20, 15 20, 10 10))
  - B - POLYGON ((20 10, 30 0, 40 10, 30 20, 20 10))



Example: Intersection Components



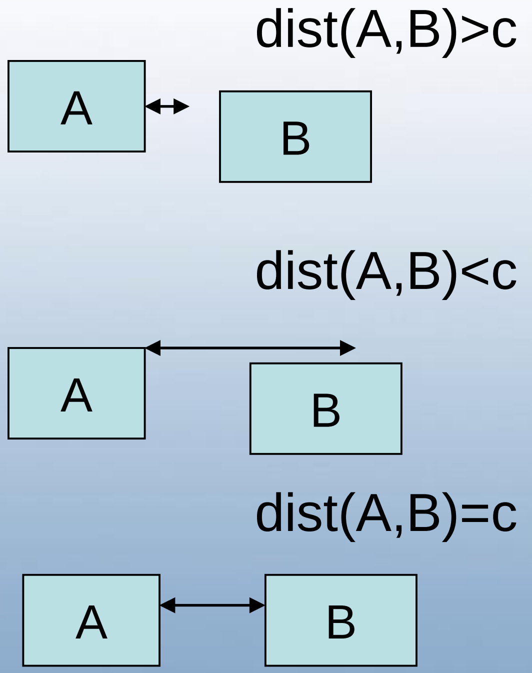
Example: Intersection Component Dimensions

	i(B)	b(B)	e(B)
i(A)	2	1	2
b(A)	1	0	1
e(A)	2	1	2

Distance-based Metric Relations

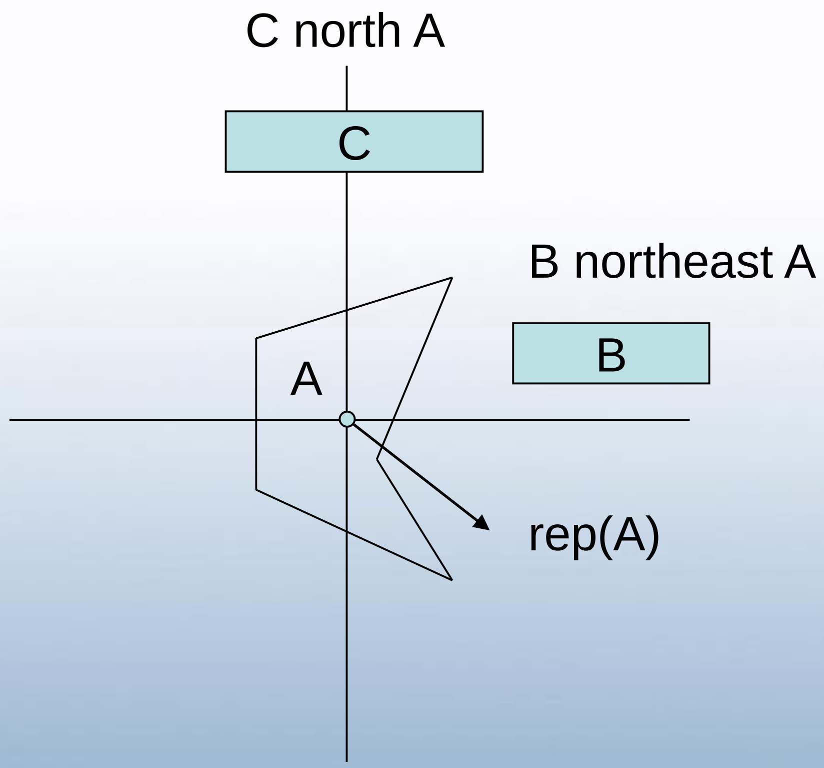
- Distance between two point sets is defined as minimal distance between their points
- If **dist** is a distance function and **c** is some real number, possible types of relations:

1. dist(A,B)>c,
2. dist(A,B)<c and
3. dist(A,B)=c



Direction Relations

- If directions of B and C are required with respect to A
- Define a representative point, rep(A)
- rep(A) defines the origin of a virtual coordinate system
- The quadrants and half planes define the direction relations
- B can have two values {northeast, east}
- Exact direction relation is northeast



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