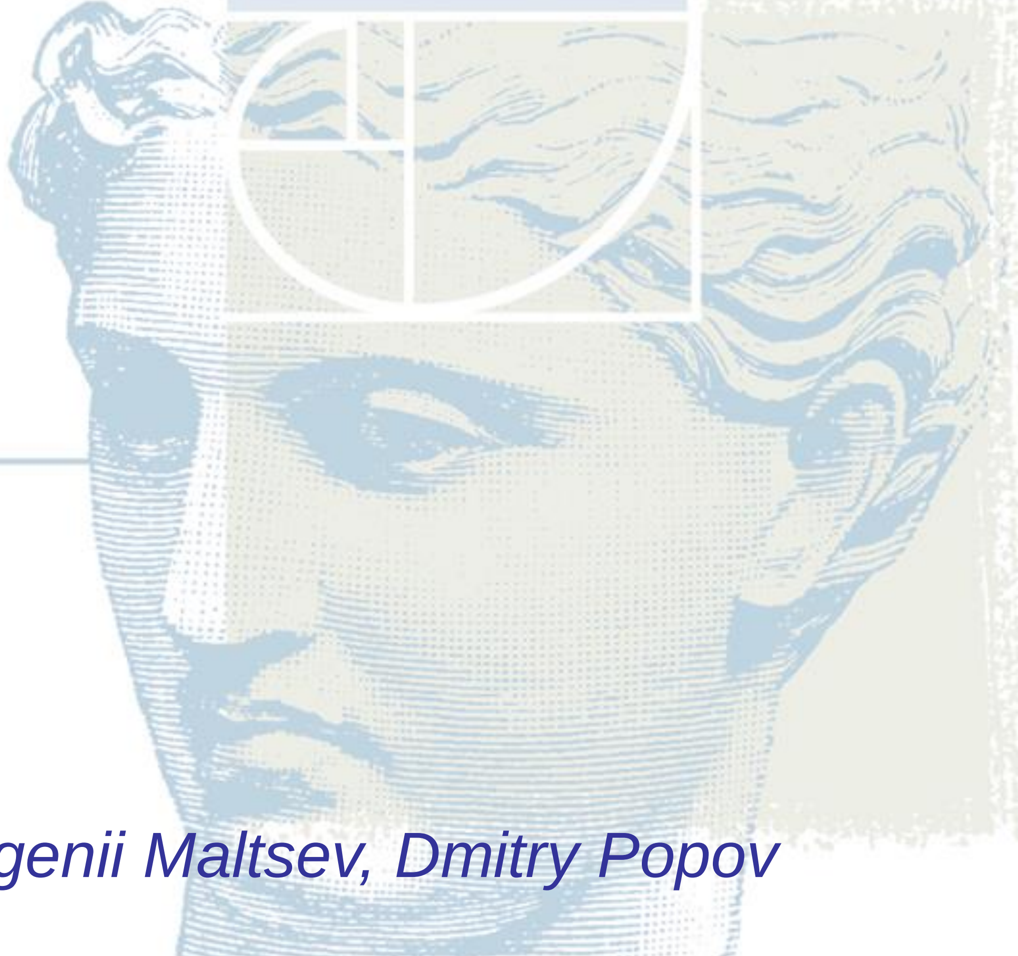
Discrete Mathematics



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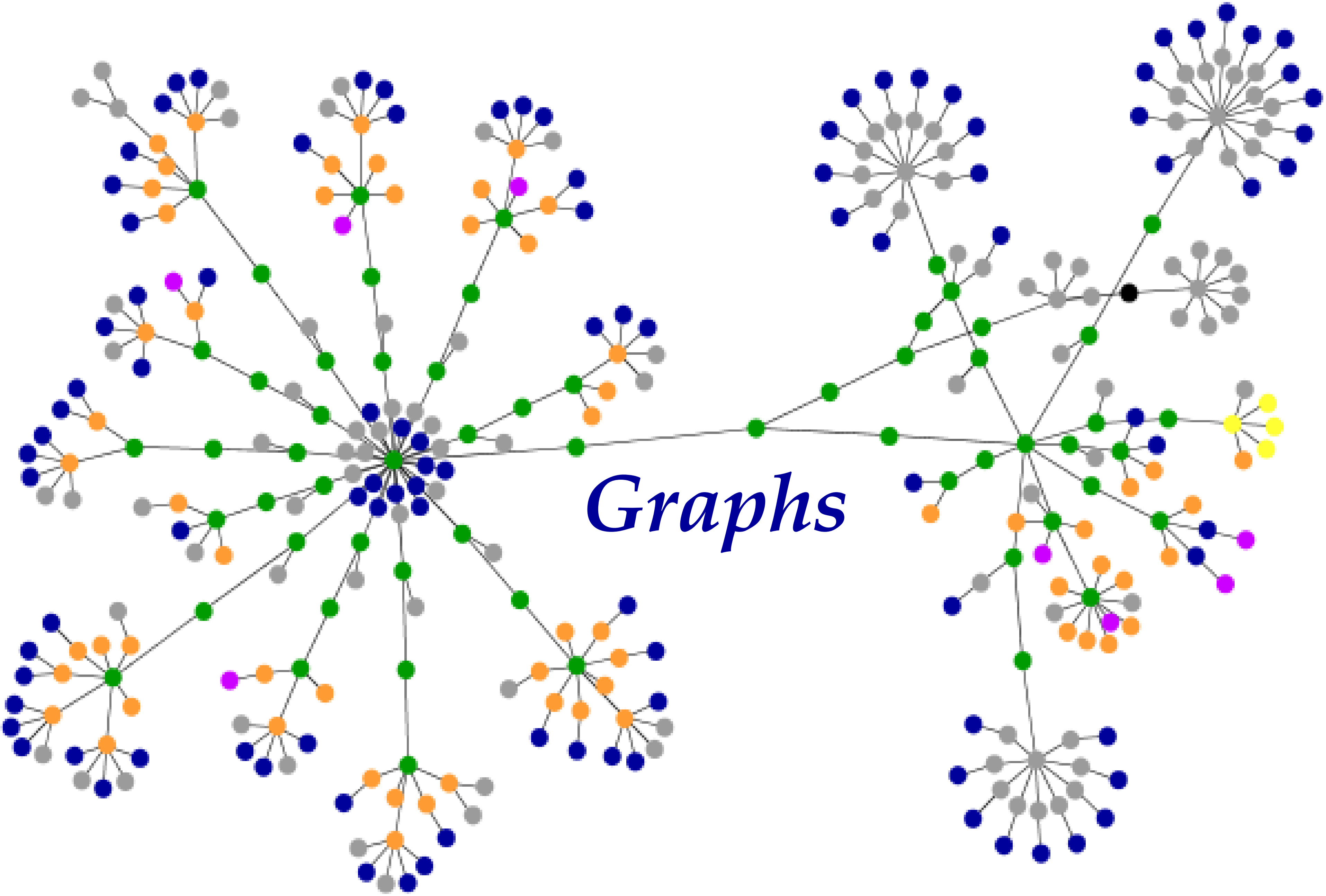
www.pasko.org/ap/DM



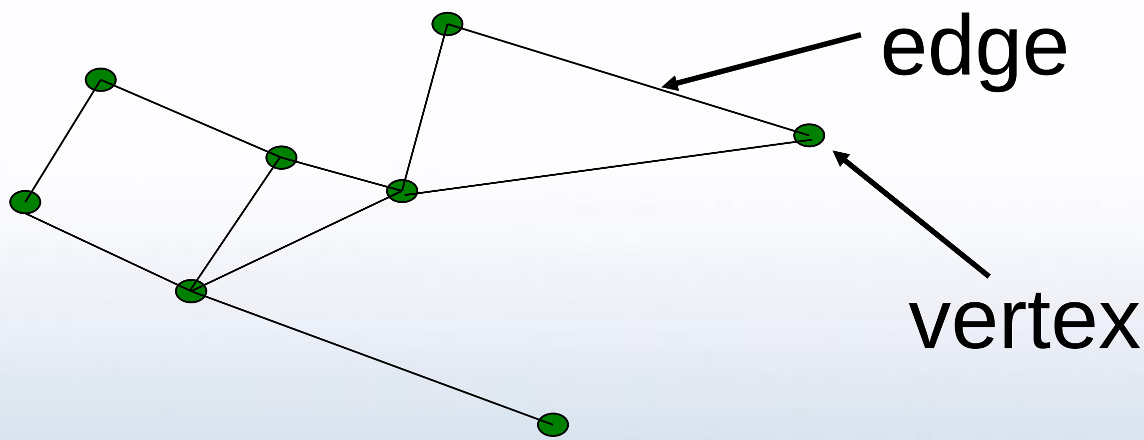
Graphs

Image www.allthingsdistributed.com

Contents

- Graph terminology
- Handshaking theorem
- Special graphs
- Graph representations

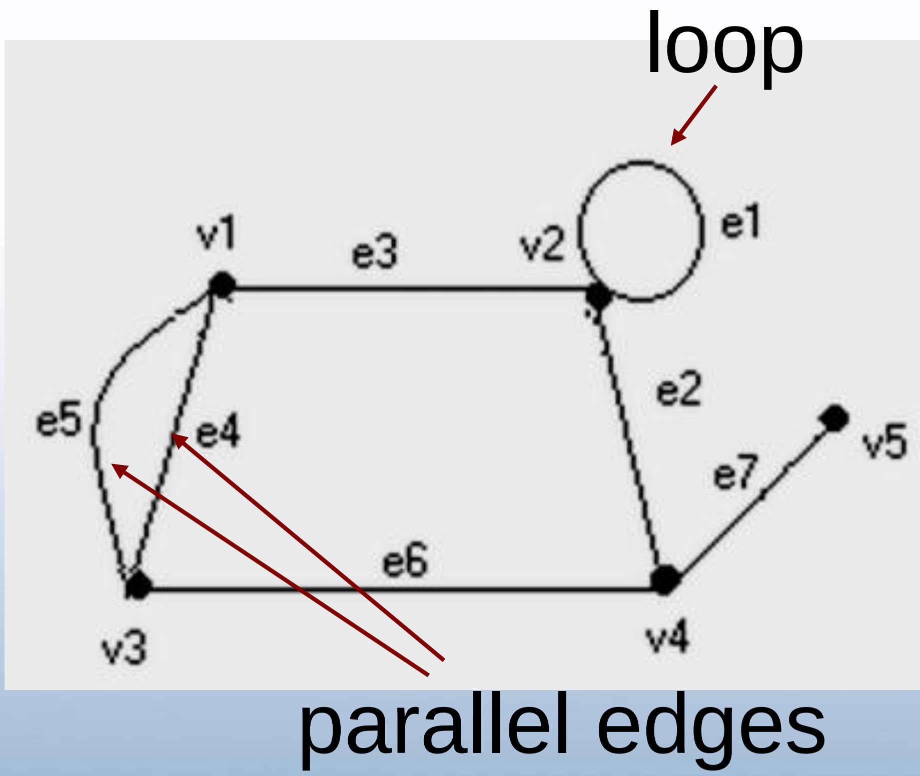
Notion of Graph



- Technical meaning of graphs in discrete mathematics is a particular class of discrete structures that is useful for representing relations between elements of sets.
- Graph has a convenient graphical representation with vertices (for elements) connected by edges (for relations).

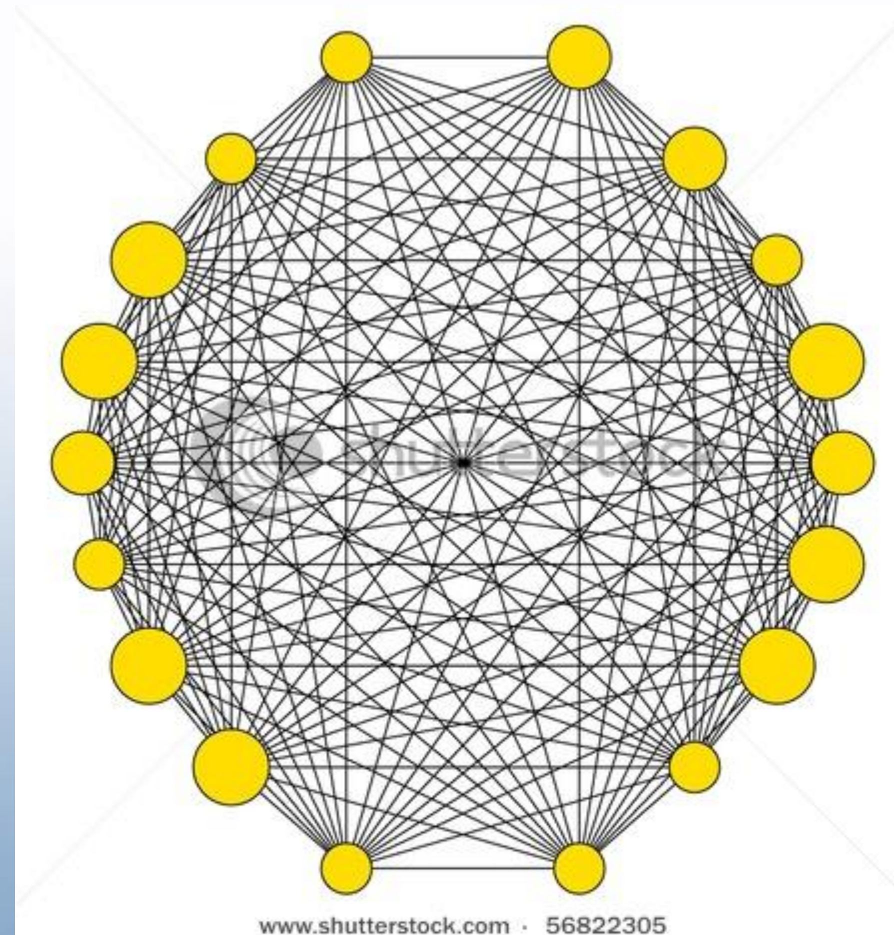
General Graph

- Graph $G = (V, E)$ consists of vertices $V = \{v1, v2, \ldots\}$ edges $E = \{e1, e2, \ldots\}$
- Each edge e_k in E is identified with an unordered pair (v_i, v_j) of vertices called the end vertices of e_k.



Relations and Graphs

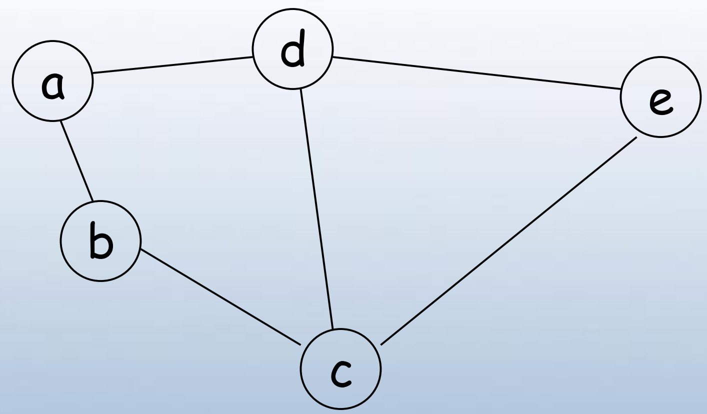
- Relation is a subset of the Cartesian product $R \subseteq A \times A$
- Cartesian product $A \times A$ includes all pairs (a,b) of elements of A: the graph of it connects all the nodes with each other with edges.
- Graph of relation R includes some subset of edges $E \subseteq V \times V$



What is missing in this graph for $A \times A$?

Simple Graph

Graph that has neither self-loop nor parallel edges is called a simple graph



G = (V,E)

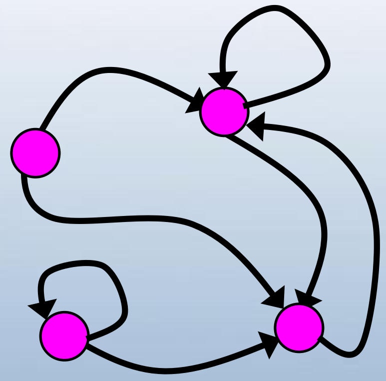
- V = {a,b,c,d,e}
- E = {(a,b),(a,d),(b,c),(c,d),(c,e),(d,e)}

Simple Graph

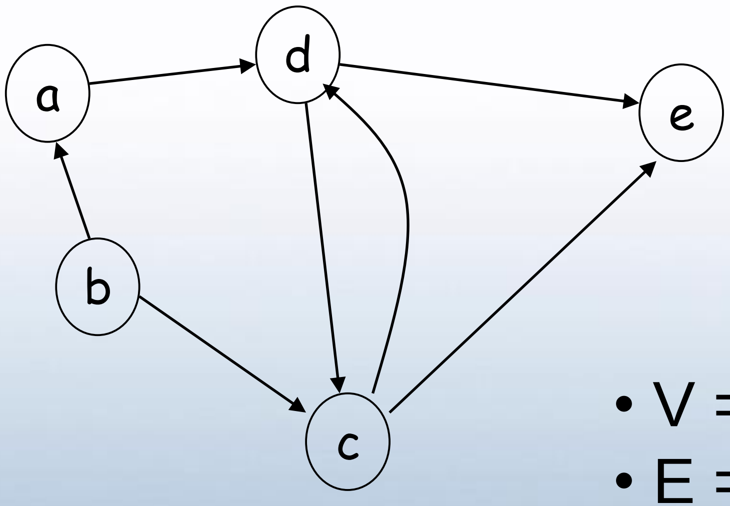
- In terms of relations, simple graph $G=(V,E)$ includes:
  - set of vertices V corresponds to the universe of the relation R
  - a set E of *edges* represents unordered pairs of distinct elements $a,b \in V$, such that aRb
  - simple graph corresponds to binary relation R which is
  - symmetric $\forall a,b((a,b)\in R \Leftrightarrow (b,a)\in R)$
  - irreflexive $\forall a\in A(\neg aRa)$

Directed Graphs

- Directed graph or a digraph (V,E) consists of a set of vertices V and a set E of *ordered pairs* of vertices.
- Example: V = people, E={(x,y) | x loves y}



Example 1



- G = (V,E)
  - V is set of vertices
  - E is set of directed edges

- V = {a,b,c,d,e}
- E = {(a,d),(b,a),(b,c),(c,d),(c,e),(d,c),(d,e)}

Adjacency

Let *G* be an undirected graph with edge set *E*. Let $e \in E$ be (or map to) the pair (u,v). Then we say:

- Vertices *u*, *v* are *adjacent* or *connected*.
- Edge *e* is *incident with* vertices *u* and *v*.
- Edge *e connects u* and *v*.
- Vertices *u* and *v* are *endpoints* of edge *e*.

Degree of a Vertex

- Let *G* be an undirected graph, $v \in V$ a vertex.
- The *degree* of *v*, deg(*v*), is its number of incident edges (except that any self-loops are counted twice)
- A vertex with degree 0 is called *isolated vertex*
- A vertex with degree 1 is called a *leaf vertex* or *end vertex*, and the edge incident with that vertex is called a *pendant edge*
- Note: degree = valency

Degree of a Vertex

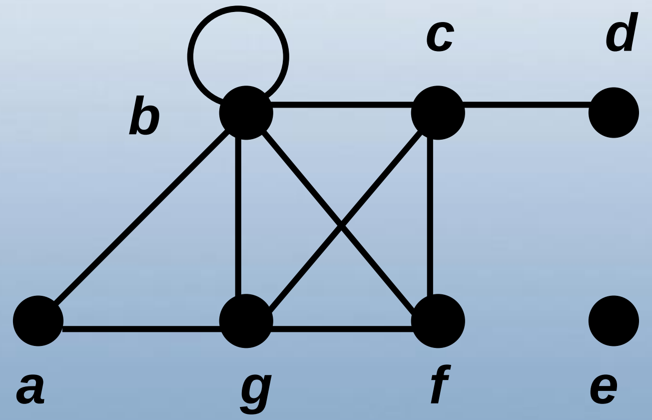
Find the degree of all the other vertices.

deg(*a*) = 2   deg(c) = 4   deg(f) = 3   deg(*g*) = 4

TOTAL of degrees = 2 + 4 + 3 + 4 + 6 + 1 + 0 = 20

TOTAL NUMBER OF EDGES = 10

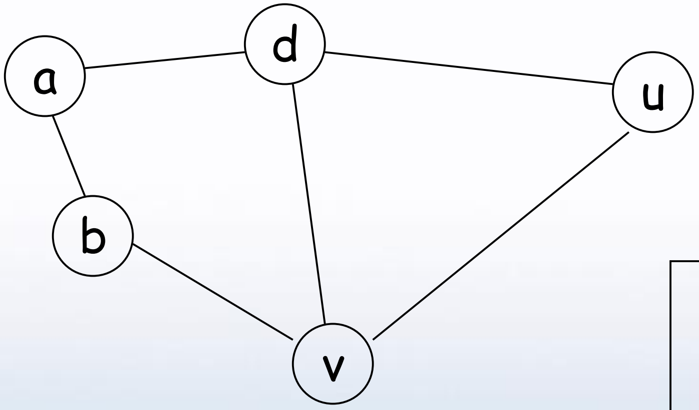
deg(*b*) = 6



deg(*d*) = 1

deg(*e*) = 0

Handshaking Theorem



G = (V,E)

$$2e = \sum_{v \in V} \deg(v)$$

For a simple graph G with e edges, the sum of the degrees is 2e

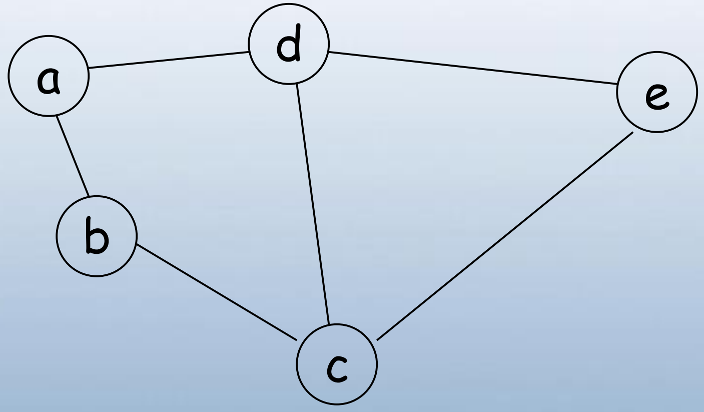
Why?

- Edge (u,v) adds 1 to the degree of vertex u and vertex v
- Therefore edge(u,v) adds 2 to the sum of the degrees of G
- Consequently the sum of the degrees of the vertices is 2e
- Note: This applies even if multiple edges and loops are present.

2e = deg(a) + deg(b) + deg(d) + deg(v) + deg(u) = 2 + 2 + 3 + 3 + 2 = 12

Handshaking Theorem

There is an even number of vertices of odd degree

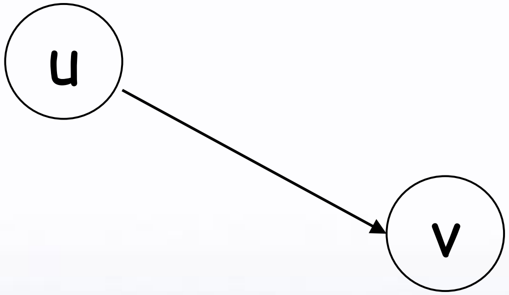


deg(d) = 3 and deg(c) = 3

Directed Adjacency

- Let *G* be a directed graph, and let *e* be an edge of *G* that is (*u*,*v*). Then we say:
  - – *u* is *adjacent to v*, *v* is *adjacent from u*
  - – *e comes from* u, e *goes to* v.
  - – *e connects u to v*, *e goes from u to v*
  - – the *initial vertex* of *e* is *u*
  - – the *terminal vertex* of *e* is *v*

Directed Degrees



- (u,v) is a directed edge
- u is the initial vertex
- v is the terminal or end vertex

In-degree of a vertex - number of edges with v as terminal vertex

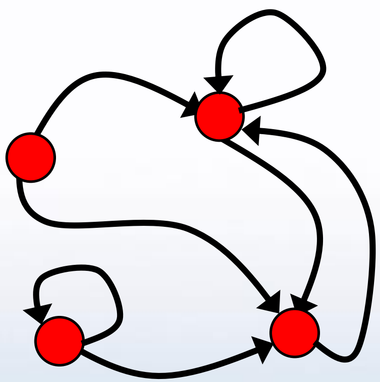
$$\deg^{+}(v)$$

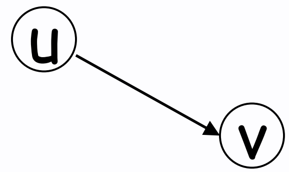
Out-degree of a vertex - number of edges with v as initial vertex

$$\deg^{-}(v)$$

$$\deg(v) \equiv \deg^{-}(v) + \deg^{+}(v)$$

Directed Handshaking Theorem





- (u,v) is a directed edge
- u is the initial vertex
- v is the terminal or end vertex

$$\sum_{v \in V} \deg^{+}(v) = \sum_{v \in V} \deg^{-}(v) = |E|$$

Total in-degrees is equal to total out-degrees and equal to the number of edges.
Each directed edge (u,v) adds 1 to the out-degree of one vertex and adds 1 to the in-degree of another.

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- Graph representations
- Isomorphism

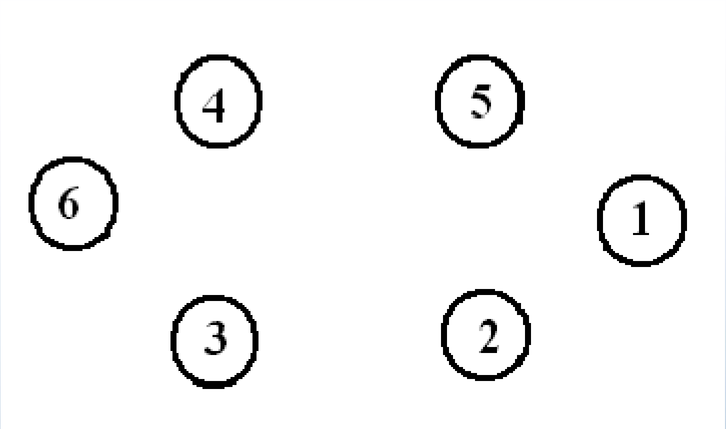
Special Graph Structures

Special cases of undirected graph structures:

- Empty and null graphs
- Complete graphs K_n
- Cycles C_n
- Wheels W_n
- *n*-Cubes Q_n
- Bipartite graphs
- Complete bipartite graphs $K_{m,n}$
- Star graphs S_k

Empty and Null Graphs

- Empty Graph / Edgeless graph
  - No edges



- Null graph
  - No nodes
  - Obviously no edges

Complete Graphs

Complete graph K_n (from the German *komplett*) or a *clique* is a graph such that for every two vertices, there exists an edge connecting the two: every vertex is connected to every other vertex.

How many edges are there in K_n?

What is the degree of every vertex?

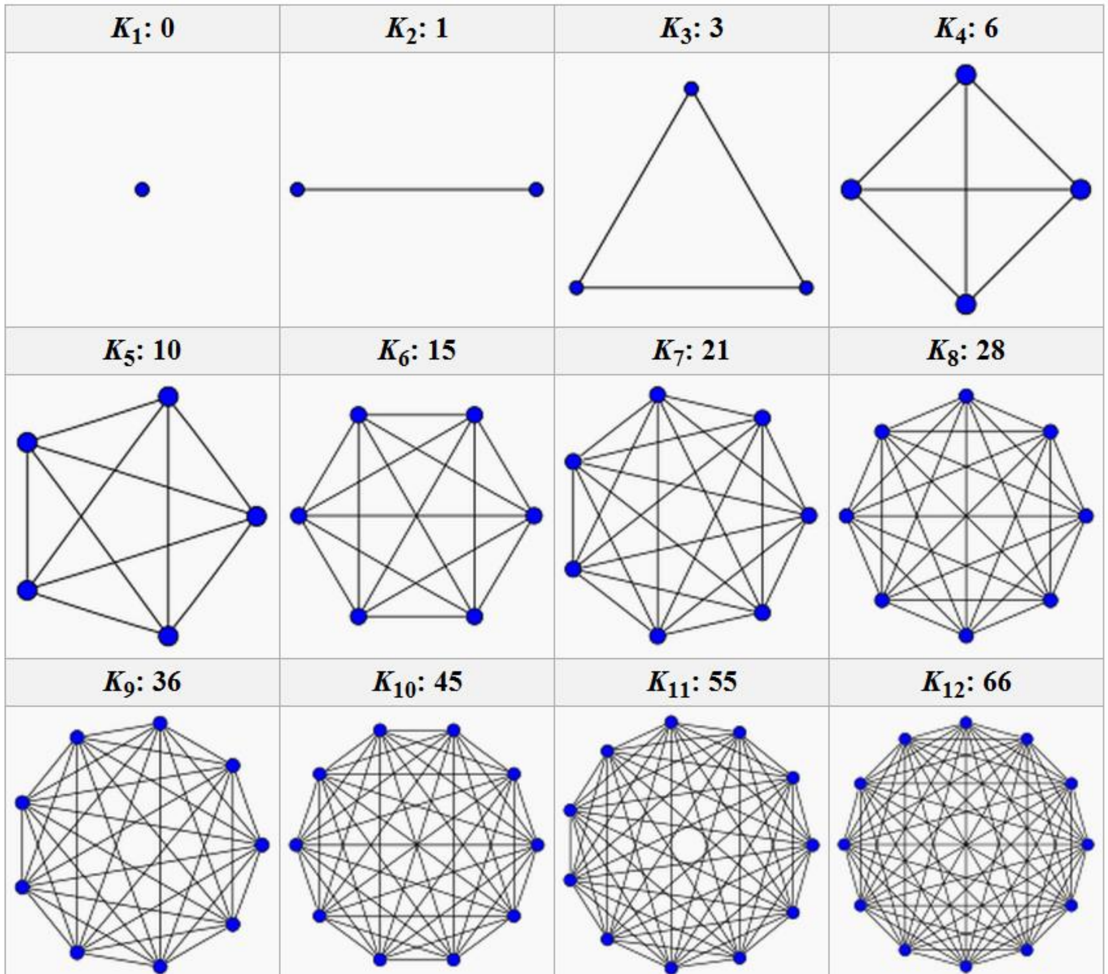
$$G = (V, E)$$

$$n = |V|$$

$$|E| = \frac{n(n-1)}{2}$$

Complete Graphs

Complete graphs on *n* vertices shown with the numbers of edges



Complete Graphs

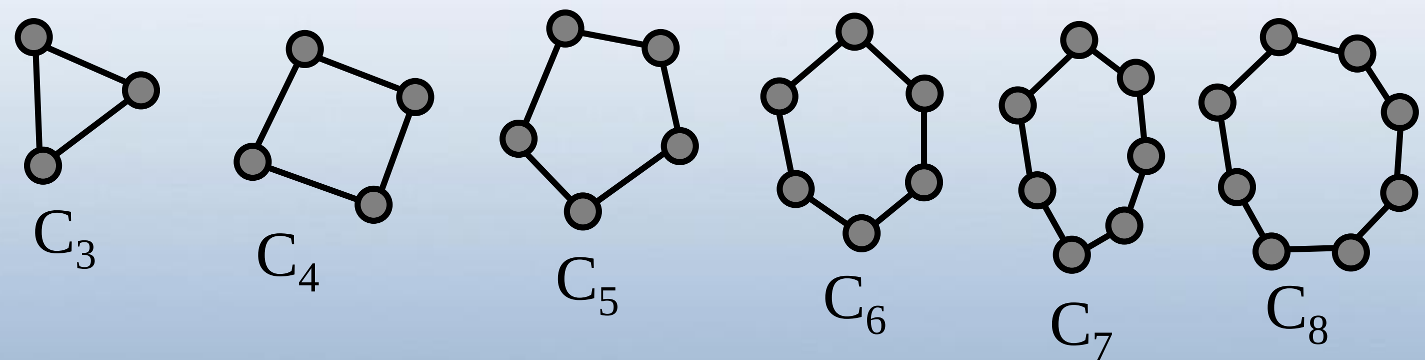
- In a complete graph every vertex is adjacent to every other vertex: $\forall u,v \in V: u \neq v \leftrightarrow (u,v) \in E$
- Can E contain any edges connecting a vertex in V to itself (loops)?

No: this would mean $(u,v) \in E$, where $u=v$

hence $\neg\forall u,v \in V: u \neq v \leftrightarrow (u,v) \in E$.

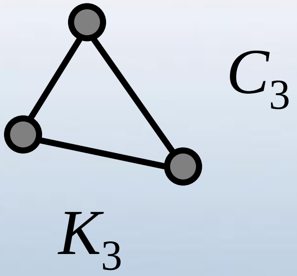
Cycles

- For any $n \geq 3$, a *cycle* on *n* vertices C_n is a simple graph where $V=\{v_1,v_2,\ldots,v_n\}$ and $E=\{(v_1,v_2),(v_2,v_3),\ldots,(v_{n-1},v_n),(v_n,v_1)\}$



How many edges are there in C_n?
What is the degree of every vertex?

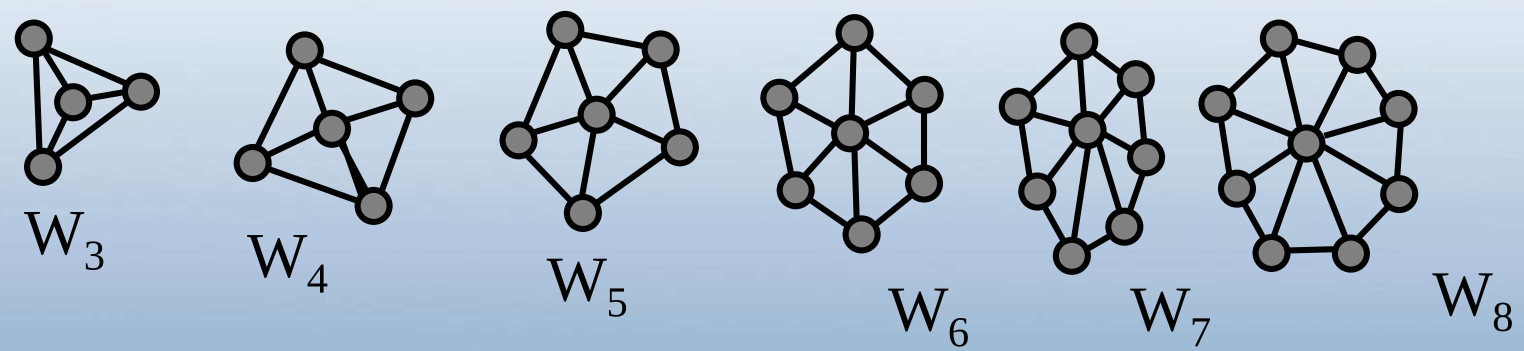
- Can a cycle be a complete graph?
- Yes: every cycle with exactly 3 elements is a complete graph.



- No other cycle can be a complete graph.

Wheels

- For any $n \geq 3$, a *wheel* W_n, is a simple graph obtained by taking the cycle C_n and adding one extra vertex v_{hub} and n extra edges $\{(v_{\text{hub}},v_1), (v_{\text{hub}},v_2),\ldots,(v_{\text{hub}},v_n)\}$.

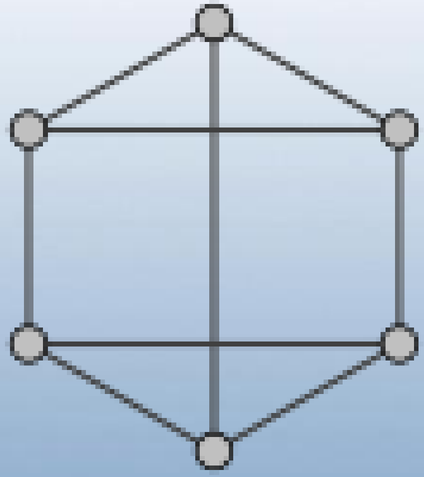


How many edges are there in W_n?

What is the degree of every vertex?

Regular Graphs

A graph is n-regular if every vertex has the same degree n



Example:

$\forall v \; deg(v) = 3$

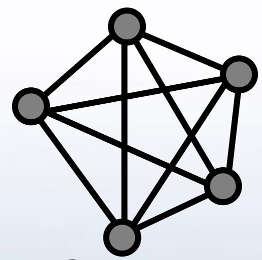
3-regular graph

Regular Graphs

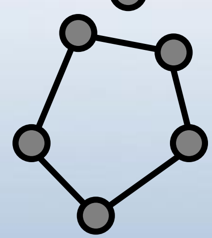
Which of these graphs are regular?

What degree?

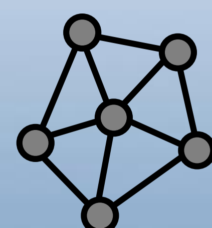
- – Complete graphs?



- – Cycle graphs?



- – Wheel graphs?

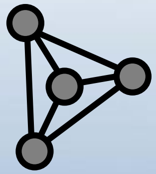


Regular Graphs

Which of these are regular?

What degree?

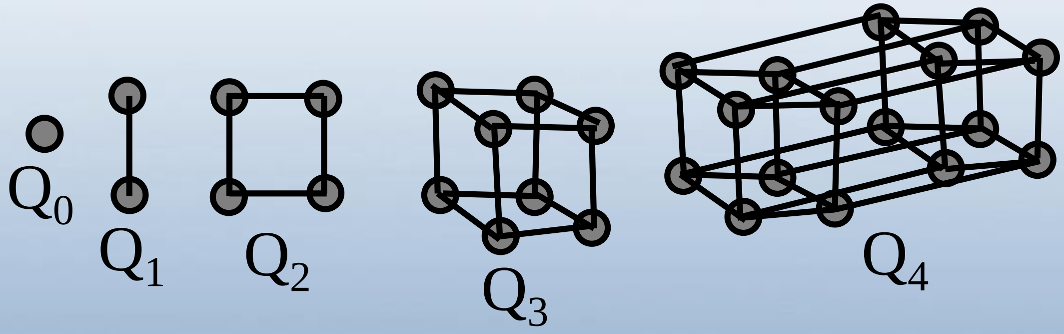
- – Complete graphs? Yes: degree n-1 (for n nodes)
- – Cycle graphs? Yes: degree 2
- – Wheel graphs? No, except W_3



W_3

n-Cubes

- For any $n \in \mathbf{N}$, the n-cube or hypercube Q_n is a simple graph consisting of two copies of Q_{n-1} connected together at corresponding nodes. Q_0 has 1 node.



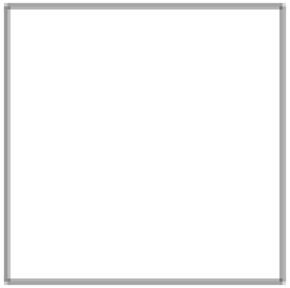
Number of vertices: 2^n

Number of edges: $n2^{n-1}$

n-Cubes

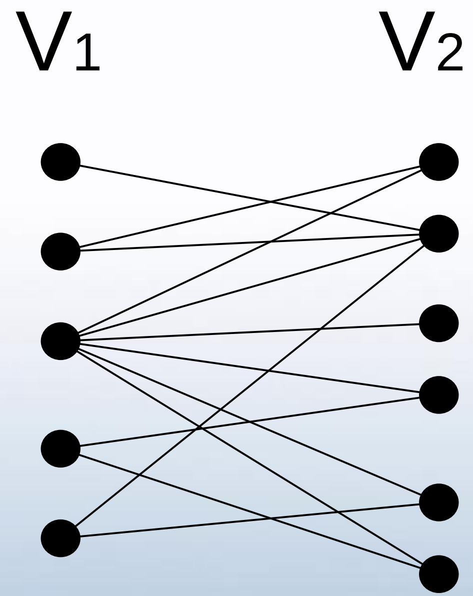
Construction of the hypecube graph Q_4:

- 2 copies of Q_2 with connected corresponding nodes = Q_3
- 2 copies of Q_3 with connected corresponding nodes = Q_4



Bipartite Graphs

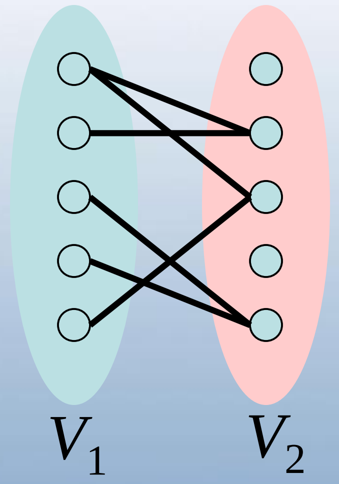
A graph $G=(V,E)$ is *bipartite* (two-part) if $V = V_1 \cup V_2$ where $V_1 \cap V_2=\emptyset$ and $\forall e \in E: \exists v_1 \in V_1, v_2 \in V_2: e=(v_1,v_2)$



The graph can be divided into two parts in such a way that all edges go between the two parts.

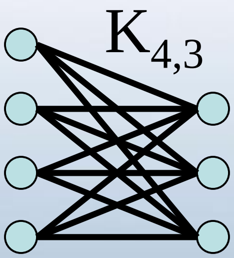
- Bipartite graphs are extremely common for modelling a domain that consists of two different kinds of entities
  - Animals in a zoo, linked with their keepers
  - Words, linked with numbers of letters in them
  - Logical formulas, linked with English sentences that express their meaning

- Given a (bipartite) graph, can there be more than one way of partitioning V into V_1 and V_2 ?
- Yes: isolated vertices can be put in either part:



Complete Bipartite Graphs

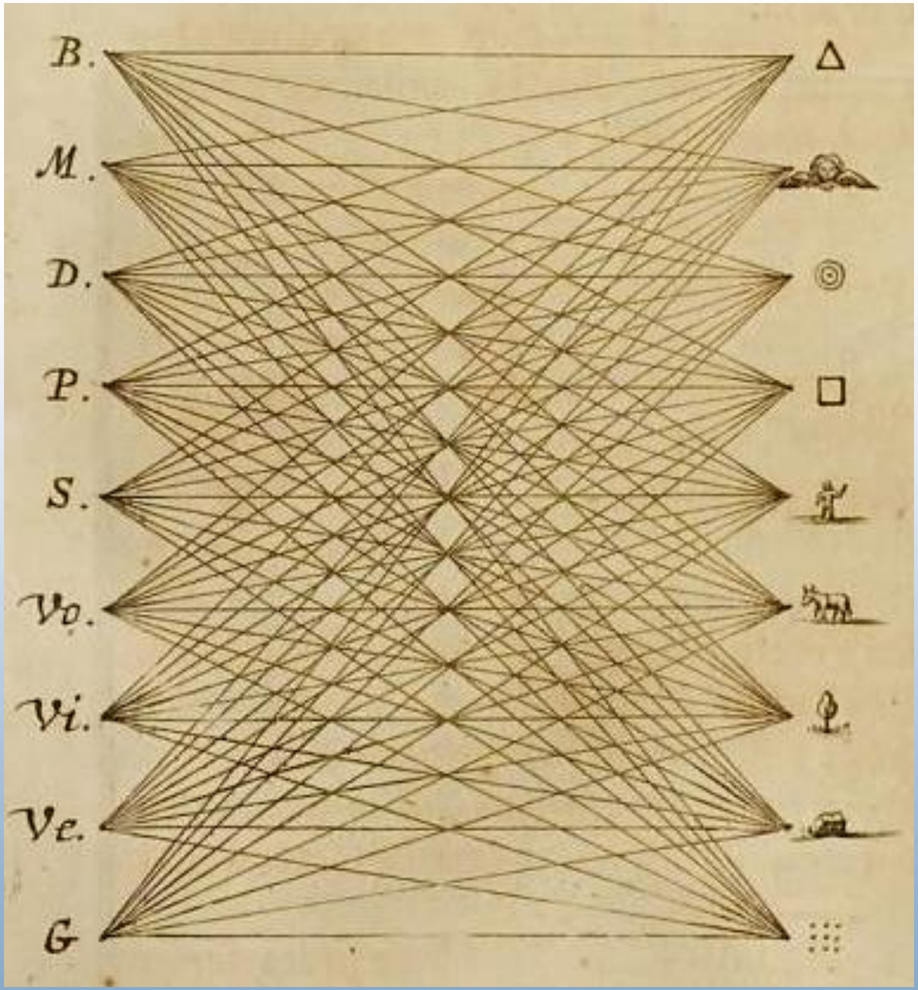
- For $m,n \in \mathbf{N}$, the *complete bipartite graph* $K_{m,n}$ is a bipartite graph where $|V_1| = m$, $|V_2| = n$, and $E = \{(v_1,v_2)|v_1 \in V_1 \wedge v_2 \in V_2\}$.
  - That is, there are *m* nodes in the left part, *n* nodes in the right part, and every node in the left part is connected to every node in the right part.



$K_{m,n}$ has ______ nodes and ______ edges.

Bipartite graphs

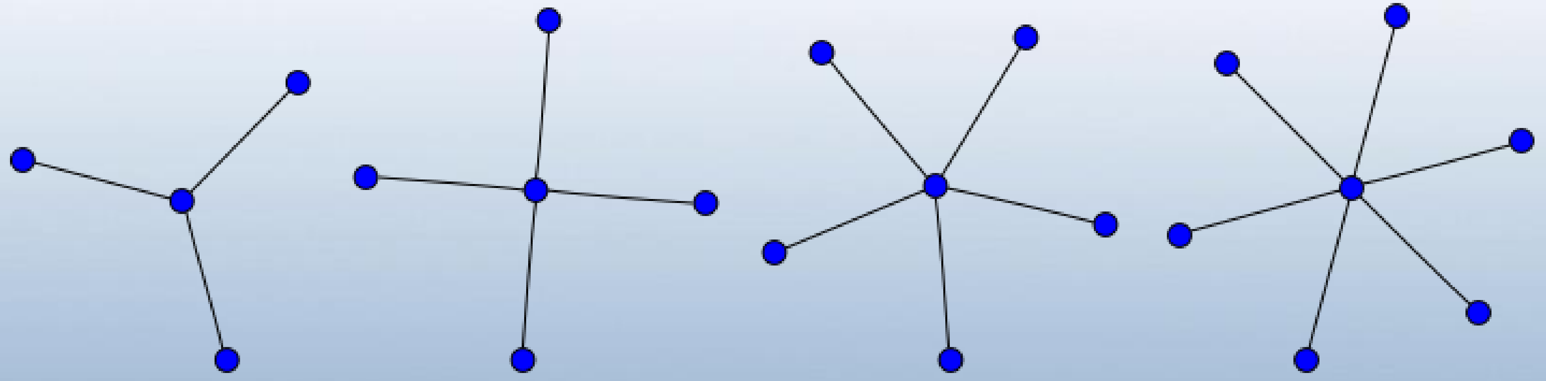
- The Cartesian product of universal subjects and absolute principles, from Athanasius Kircher's "*Ars Magna Sciendi*", 1669.
- Complete bipartite graph



Subjectorum Univerſalium cum principiis abſolutis

Star Graphs

A *star graph* S_k is the complete bipartite graph $K_{1,k}$ with one internal node of degree k and k leaves.



The star graphs S_3, S_4, S_5 and S_6

Making New Graphs

We can have a subgraph

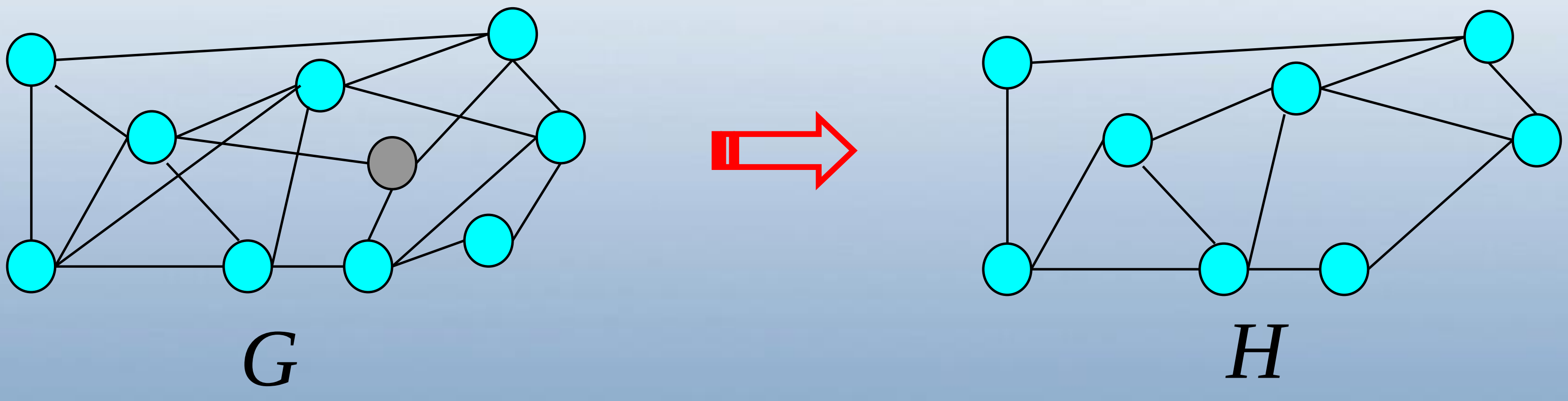
$$G = (V, E)$$
$$H = (W, F)$$
$$W \subseteq V$$
$$F \subseteq E$$

We can have a union of graphs

$$G_1 = (V_1, E_1)$$
$$G_2 = (V_2, E_2)$$
$$G_3 = G_1 \cup G_2$$
$$G_3 = (V_1 \cup V_2, E_1 \cup E_2)$$

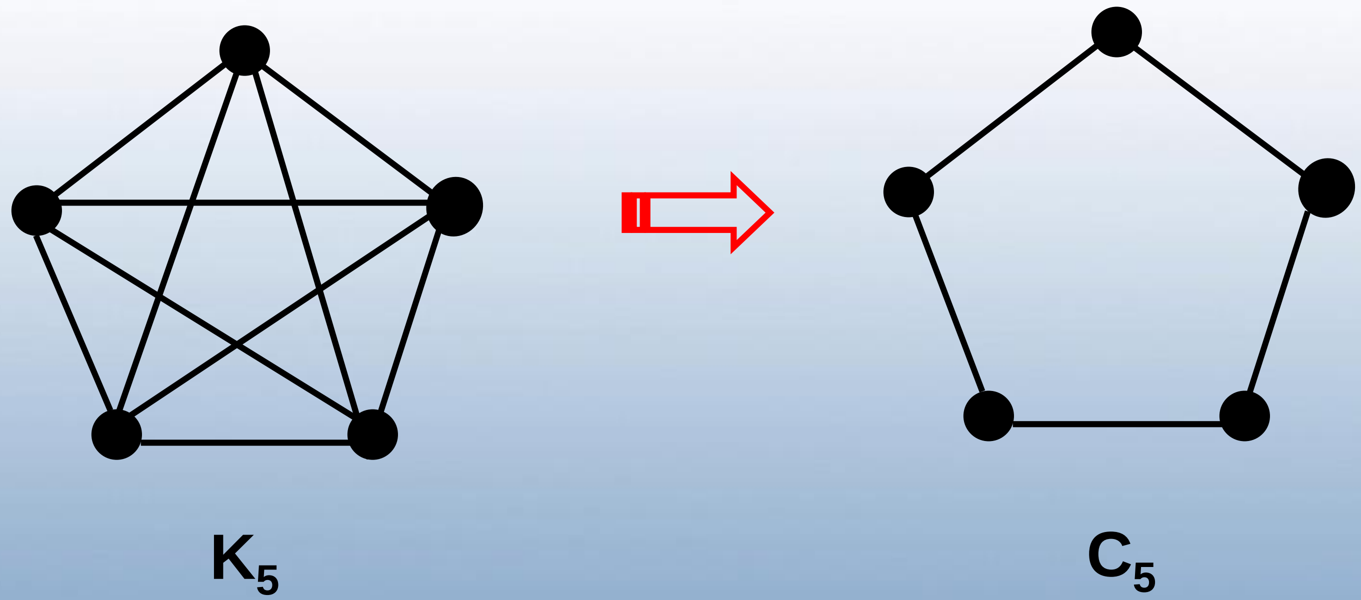
Subgraphs

Subgraph of a graph $G=(V,E)$ is a graph $H=(W,F)$ where $W \subseteq V$ and $F \subseteq E$.



Subgraphs

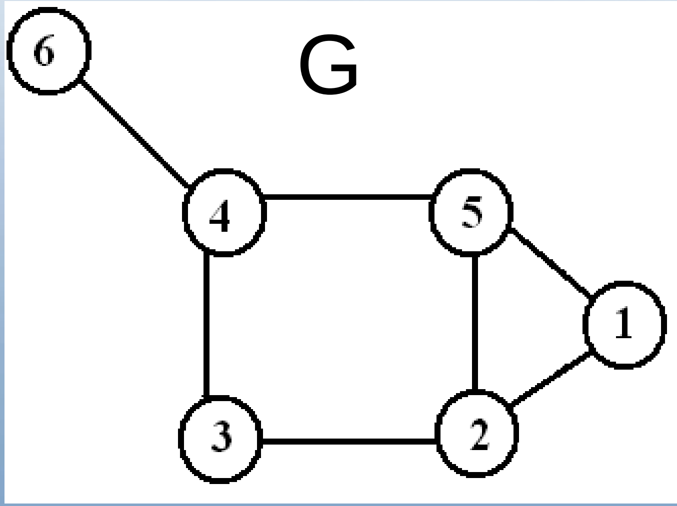
C_5 is a subgraph of K_5

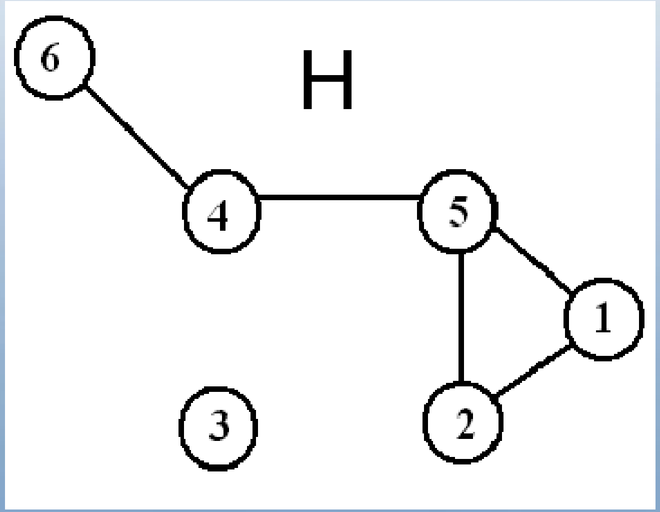


Spanning Subgraph

Spanning subgraph H has the same vertex set as graph G.

- Possibly not all the edges
- "H spans G".

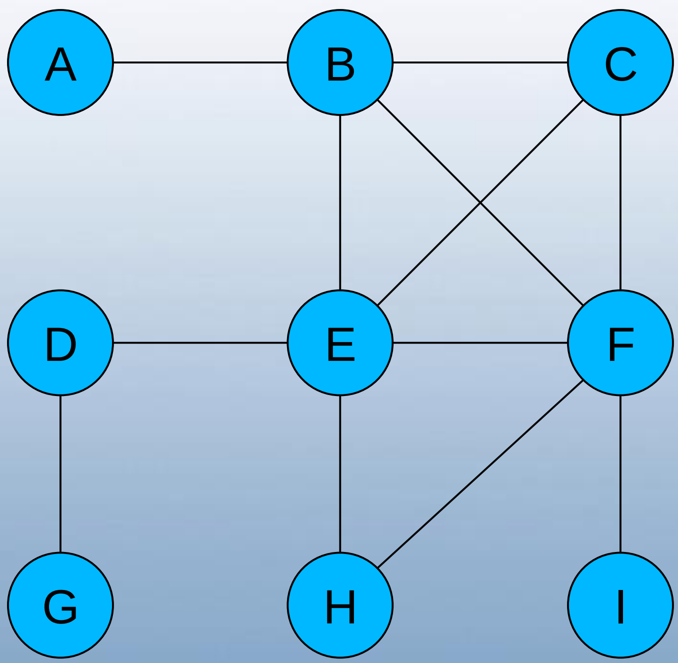


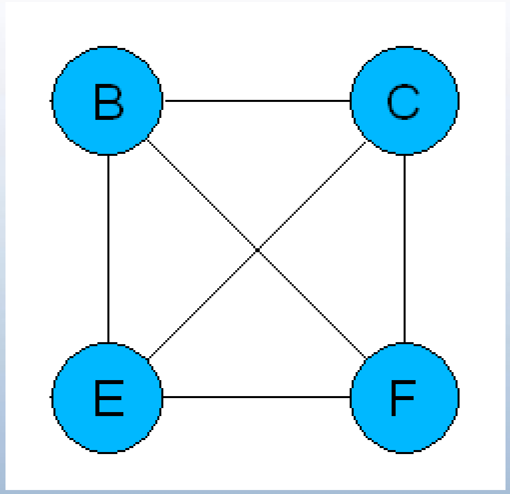


Special Subgraphs: Cliques

A clique in a graph is a subgraph such that every two vertices in it are connected by an edge.

A maximum clique is a maximum complete subgraph.

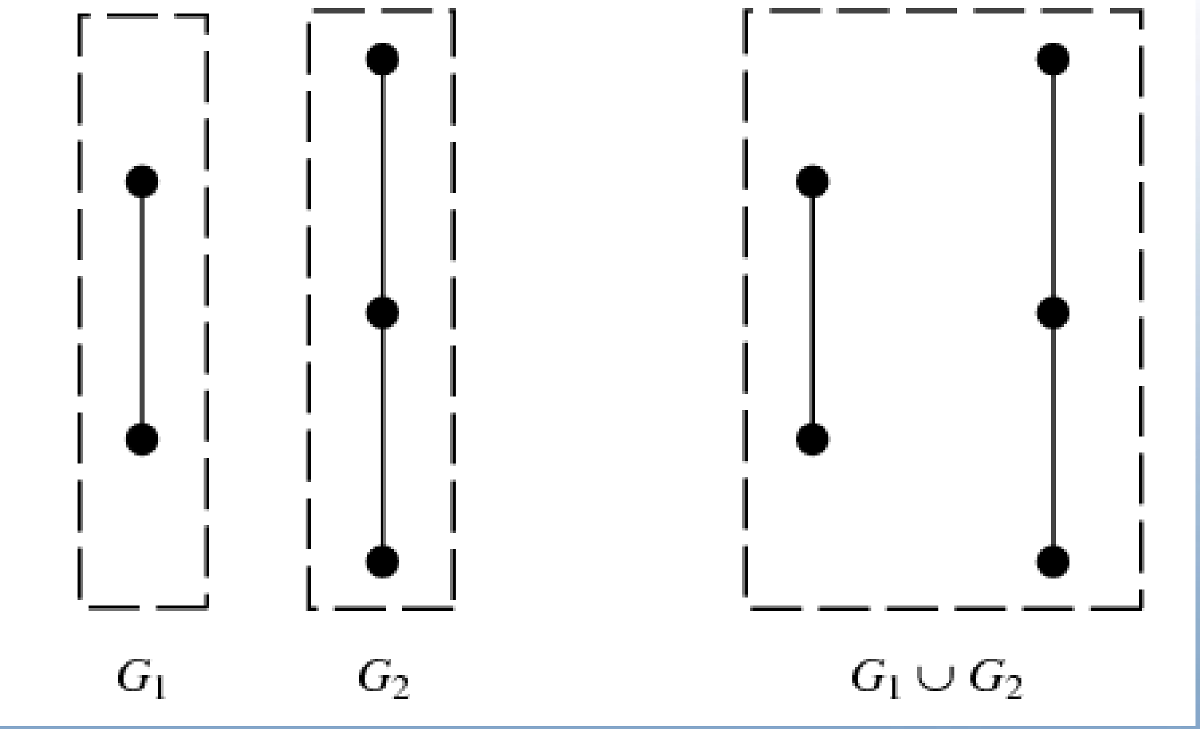




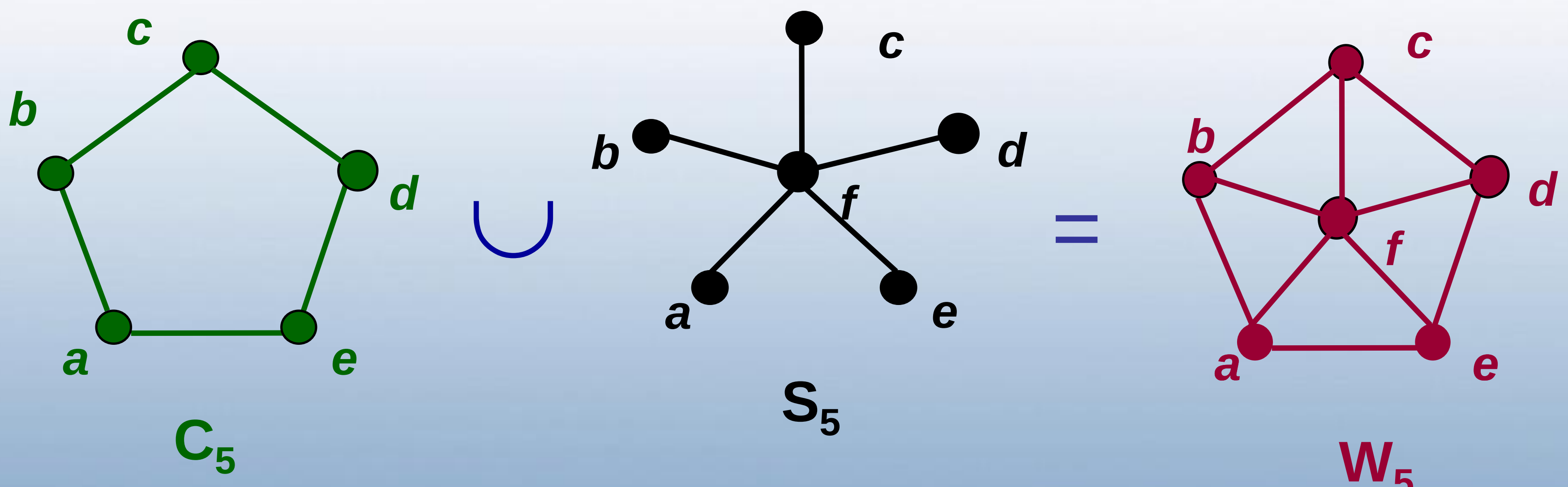
All complete graphs are their own cliques.

Graphs Union

Union $G_1 \cup G_2$ of two simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2,E_2)$ is the simple graph $(V_1 \cup V_2, E_1 \cup E_2)$.



W_5 is the union of S_5 and C_5



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- Graph representations
- Isomorphism

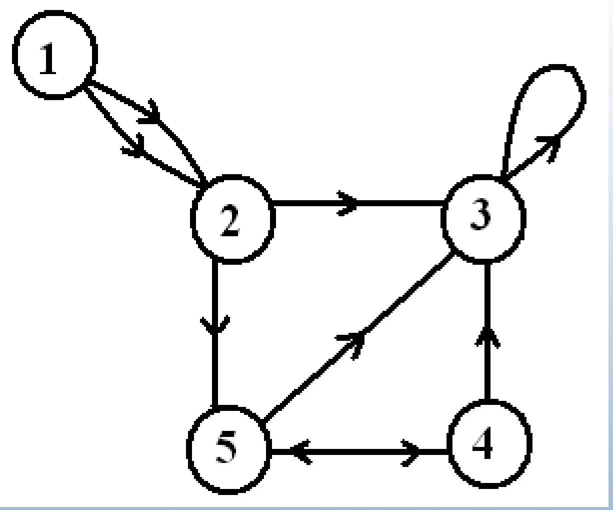
Graph Representations

Graph representations:

- Edge list
- Adjacency list
- Adjacency matrix
- Incidence matrix

Edge List

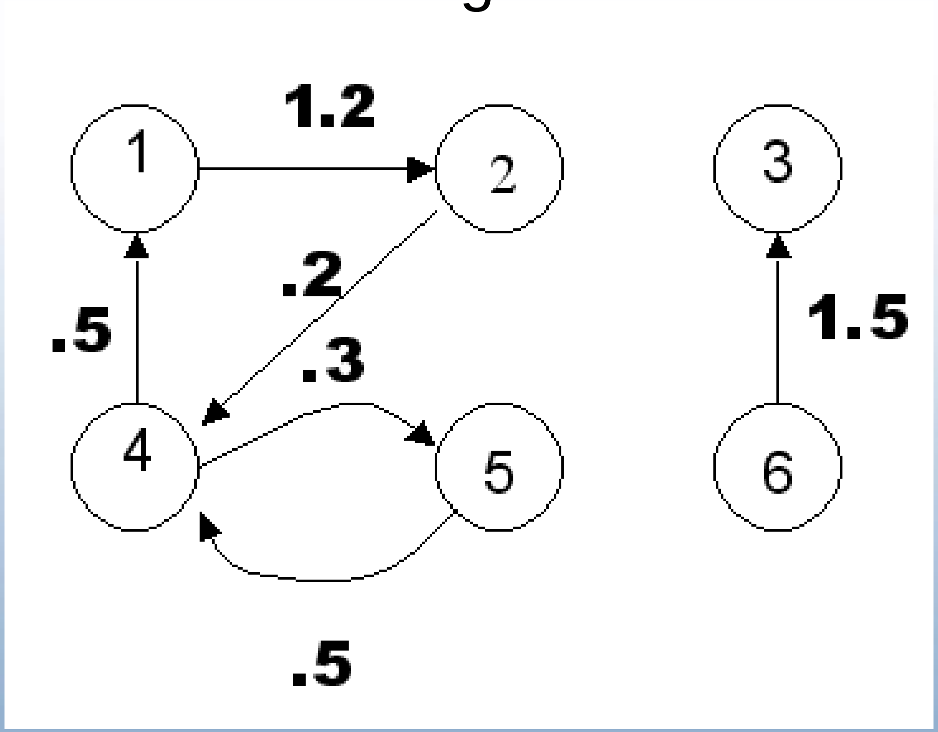
- Edge List: pairs (ordered if directed) of vertices



Edge List
1 2
1 2
2 3
2 5
3 3
4 3
4 5
5 3
5 4

Edge Lists for Weighted Graphs

- Edge List: pairs (ordered if directed) of vertices with weights and other data

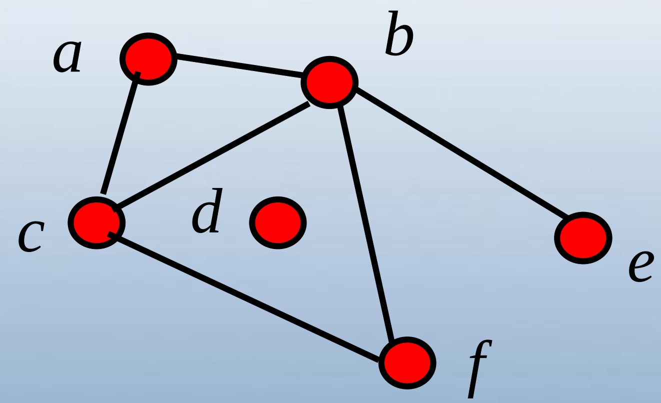


<u>Edge List</u>

1	2	1.2
2	4	0.2
4	5	0.3
4	1	0.5
5	4	0.5
6	3	1.5

Adjacency List

Table with one row per vertex, listing its adjacent vertices (node list).



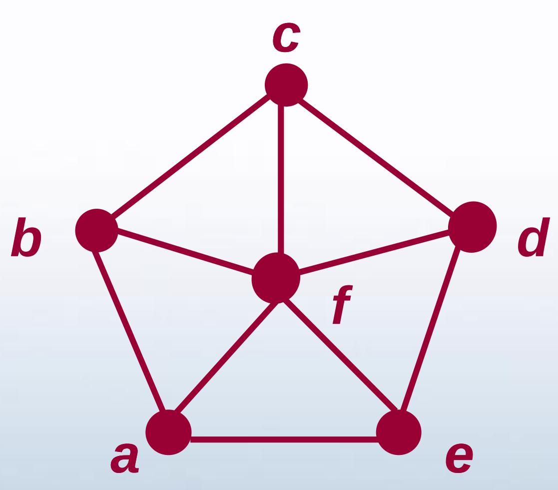
Vertex	*Adjacent Vertices*
a	*b*, *c*
b	*a*, *c*, *e*, *f*
c	*a*, *b*, *f*
d	
e	*b*
f	*c*, *b*

Adjacency Matrix

A simple graph $G = (V, E)$ with n vertices can be represented by its adjacency matrix A, where entry a_{ij} in row i and column j is

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge in G,} \\ 0 & \text{otherwise.} \end{cases}$$

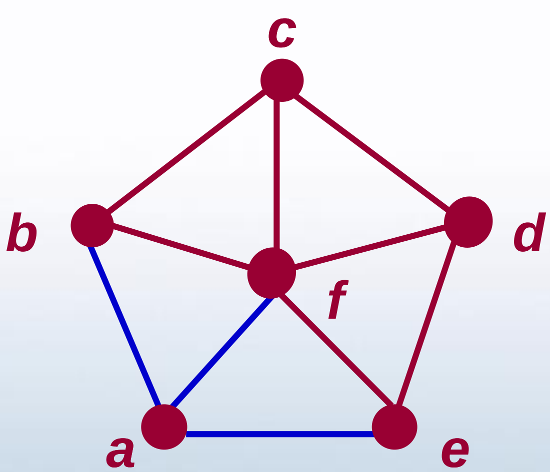
Example



W_5

This graph has 6 vertices a, b, c, d, e, f. We can arrange them in that order for both rows and columns of the matrix.

Adjacency matrix



W_5

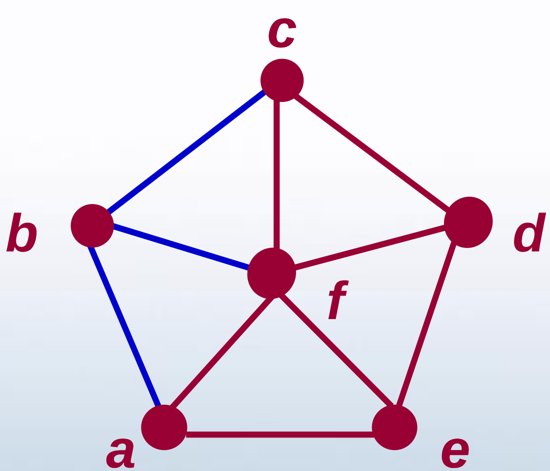
TO

FROM

	a	b	c	d	e	f
a	**0**	**1**	**0**	**0**	**1**	**1**
b						
c						
d						
e						
f						

There are edges from a to b, from a to e, and from a to f

Adjacency matrix



W_5

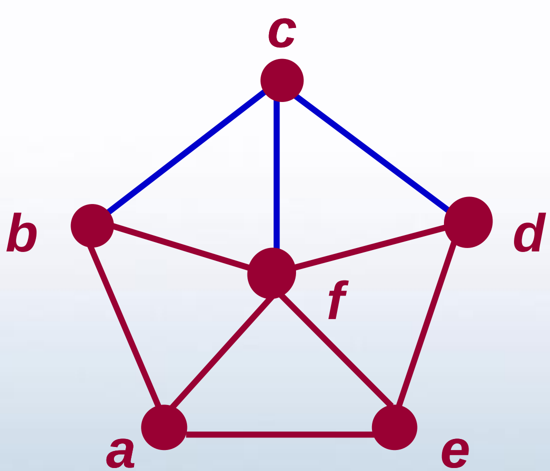
TO

FROM

	a	b	c	d	e	f
a	**0**	**1**	**0**	**0**	**1**	**1**
b	**1**	**0**	**1**	**0**	**0**	**1**
c						
d						
e						
f						

There are edges from b to a, from b to c, and from b to f

Adjacency matrix



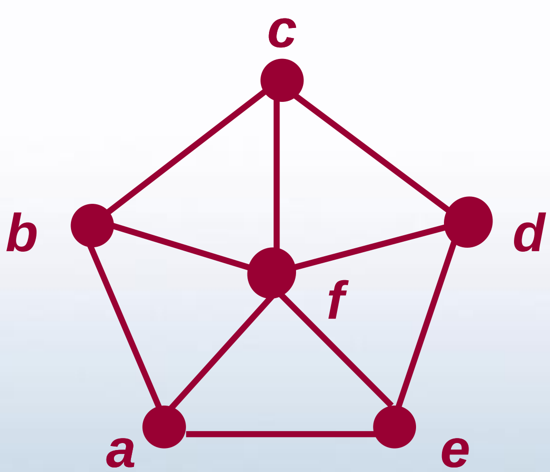
W_5

TO

FROM	a	b	c	d	e	f
a	**0**	**1**	**0**	**0**	**1**	**1**
b	**1**	**0**	**1**	**0**	**0**	**1**
c	**0**	**1**	**0**	**1**	**0**	**1**
d						
e						
f						

There are edges from c to b, from c to d, and from c to f

Adjacency matrix



W_5

TO

FROM

	a	b	c	d	e	f
a	0	1	0	0	1	1
b	1	0	1	0	0	1
c	0	1	0	1	0	1
d	0	0	1	0	1	1
e	1	0	0	1	0	1
f	1	1	1	1	1	0

Notice that this matrix is symmetric. That is $a_{ij} = a_{ji}$ Why?

Adjacency Matrix

Adjacency matrix properties:

$$(i, j) \notin E \leftrightarrow a_{i,j} = 0 \qquad (i, j) \in E \leftrightarrow a_{i,j} = 1$$

A is symmetric for simple graphs

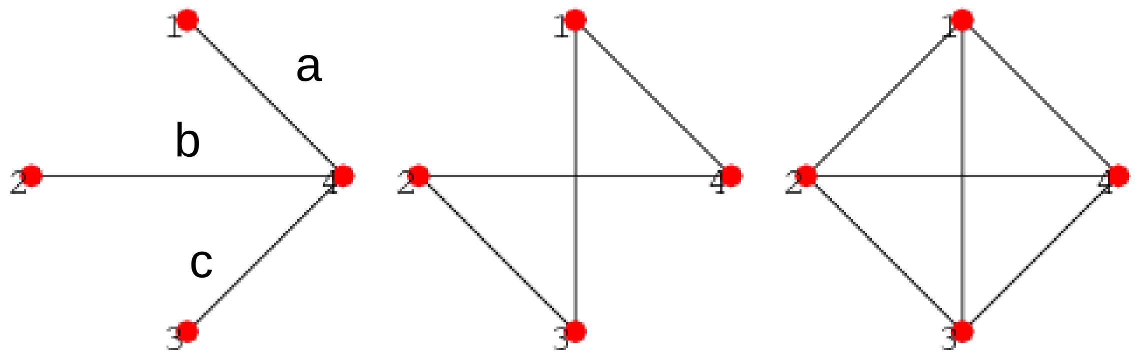
$$(i, j) \in E \leftrightarrow a_{i,j} = 1 = a_{j,i}$$

Simple graphs do not have loops (v,v)

$$\forall i (a_{i,i} = 0)$$

Incidence Matrix

The incidence matrix of a graph is a (0,1)-matrix which has a row for each vertex and column for each edge, and *(v,e) = 1* if edge *e* is incident with vertex *v*.



	a	b	c
1	1	0	0
2	0	1	0
3	0	0	1
4	1	1	1

Questions?