Implicit curves and surfaces

Algebraic objects, blobby objects, RBF

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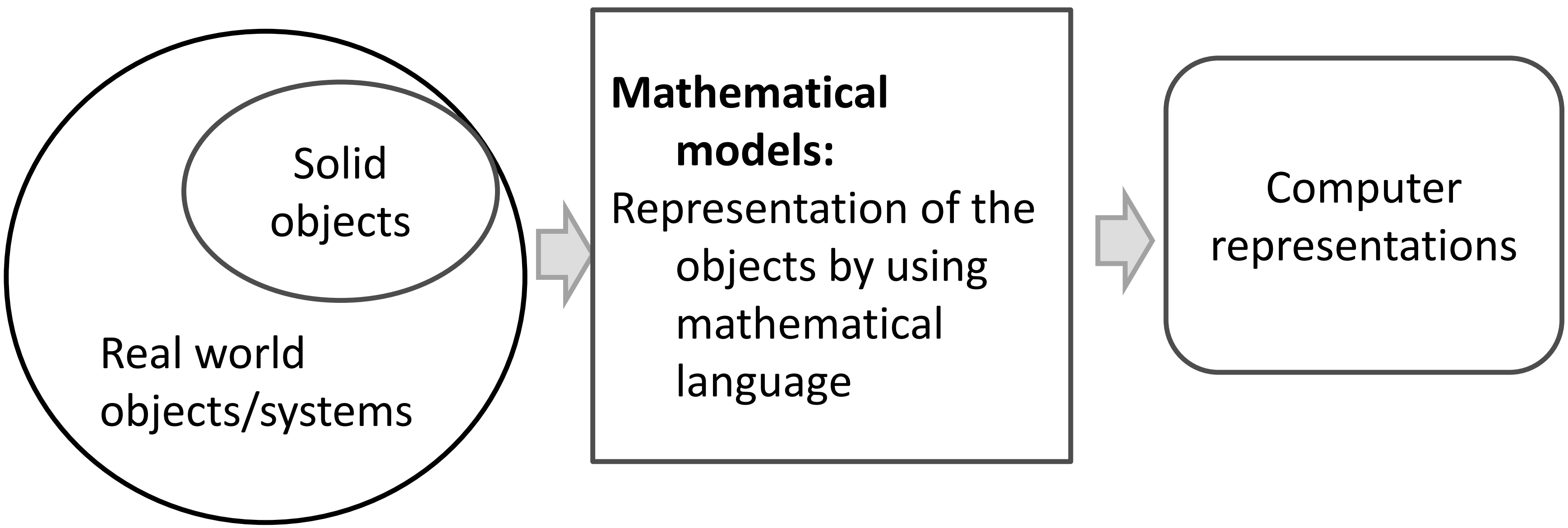
Solid modelling

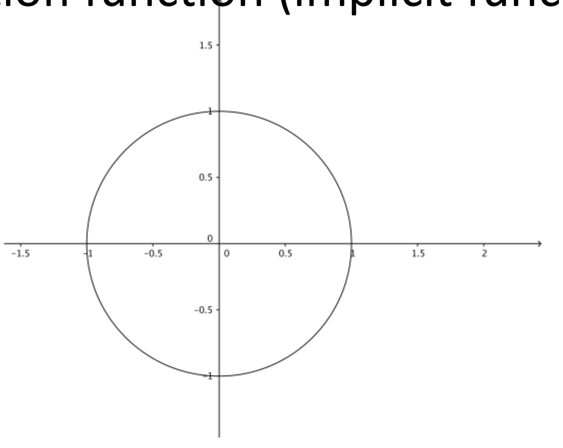


Computer representations

- 3-dimensional object is a point set
- There are 2 fundamental ways to represent a point set
  - Enumerative (and combinatorial) representations specify the rules for *generating* points in the set (and no other points). Examples:
    - Enumeration (explicit point set)
    - Groupings (voxels)
    - Cell complexes (points + neighbourhood information)
    - Parametric representation
    - Explicit representation
  - Implicit (and constructive) representation gives rules for *testing* which points belong to the set and which are not. Examples:
    - Implicit representation (some predicate can be evaluated on any point of space)
    - Constructive Solid Geometry (CSG) (constructive tree contains a limited set of solid objects and Boolean operations)

Implicit representation

- A function which is not defined explicitly, but rather is defined in terms of algebraic relationship is an implicit function
- In the implicit form the points belonging to the the object are given indirectly through a point-membership classification function (implicit function).



f(*x*, *y*) = 0 – curve on a plane

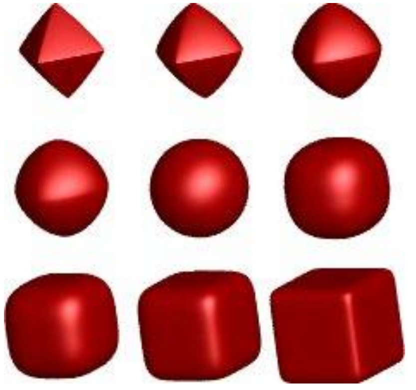
f(*x*, *y*, *z*) = 0 – surface in 3D space

- Example:

$$x^2 + y^2 = 1$$

$$x^2 + y^2 - 1 = 0$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$



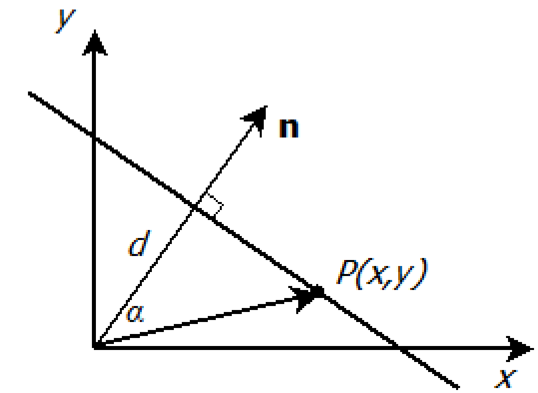
Simple objects: line and plane

- Straight line (2D)

n = (n_x, n_y) – a normal vector

p = (x, y) – position vector

$$n_x x + n_y y - d = 0$$



- Plane (3D)

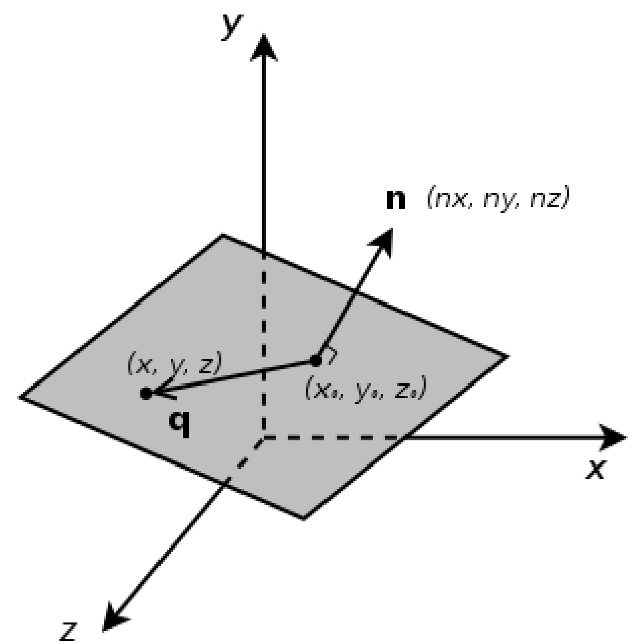
n = (n_x, n_y, n_z) – a normal vector to the plane

$\mathbf{p_o}$ = (x_o, y_o, z_o) – a point on the plane

p = (x, y, z) – position vector

$\mathbf{n} \cdot \mathbf{p} - \mathbf{n} \cdot \mathbf{p_o} = 0$

$n_x x + n_y y + n_z z - (n_x x_o + n_y y_o + n_z z_o) = 0$



Simple objects: circle and ellipse

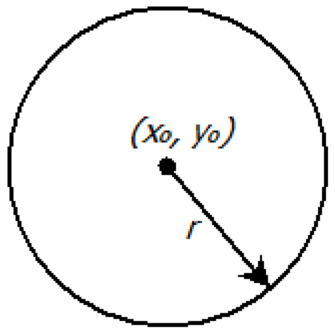
- Circle
  - Parametric form

    $x = x_0 + r \cos t$

    $y = y_0 + r \sin t$
  - Implicit form

    $(x - x_0)^2 + (y - y_0)^2 = r^2$

    $(x - x_0)^2 + (y - y_0)^2 - r^2 = 0$



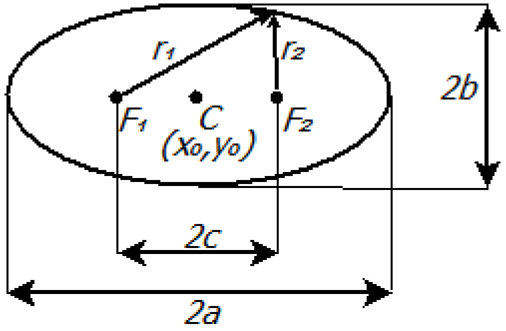
$$\frac{(x - x_0)^2}{r^2} + \frac{(y - y_0)^2}{r^2} - 1 = 0$$

- Ellipse
  - Parametric form

    $x = x_0 + a \cos t$

    $y = y_0 + b \sin t$
  - Implicit form

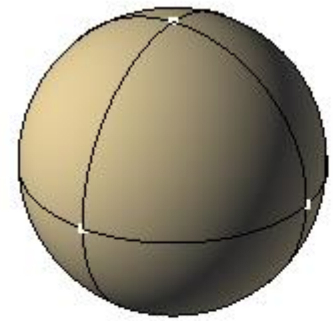
$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} - 1 = 0$$



Simple objects: sphere and ellipsoid

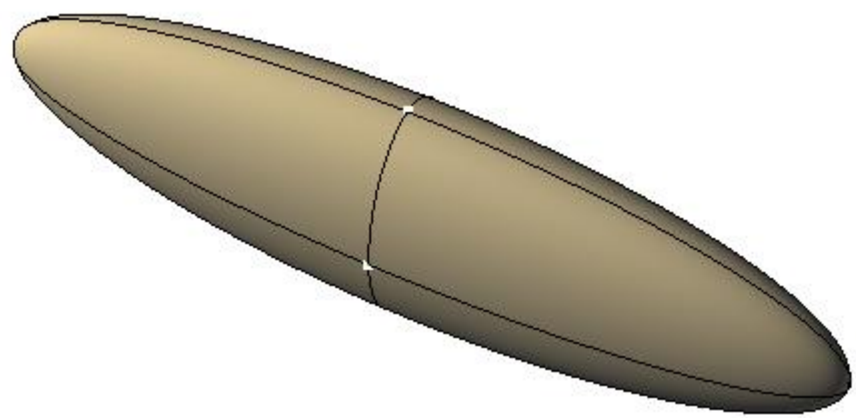
- Sphere

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2-r^2=0$$



- Ellipsoid

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}-1=0$$



Algebraic curves and surfaces

- Algebraic curve is the set of roots of an equation $f(x,y)=0$, where $f(x,y)=0$ is a polynomial in x and y
  - Coefficients of the polynomial can be defined not only over a set of real numbers, but generally over any field ***K*** (e.g. set of complex numbers)**.** In this case it is said that we have an algebraic curve over a field ***K***.
- Example:
  - straight line $n_x x + n_y y - d = 0$
  - circle $\frac{(x-x_0)^2}{r^2} + \frac{(y-y_0)^2}{r^2} - 1 = 0$
- Algebraic surface is the set of roots of a polynomial $f(x,y,z)=0$
- Example:
  - sphere $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 - r^2 = 0$

Algebraic curves and surfaces

- If the defining function is a polynomial function, we have an algebraic curve for functions on (*x*,*y*) or algebraic surface for functions on (*x*,*y*,*z*).
- Algebraic curves and surfaces are formally studied since 19th century, but many facts were known before that.

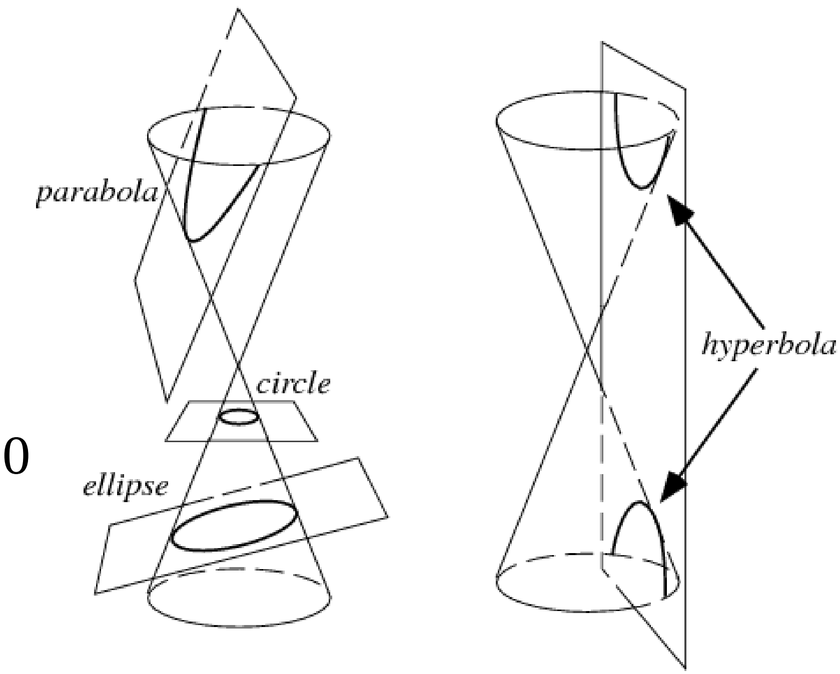
- An algebraic curve is said to be of degree $n = \max(i + j)$ where n is the maximum sum of powers of all terms $a_m x^{i_m} y^{j_m}$
  - Example: straight line $n_x x + n_y y - d = 0$, degree $n = 1$

- An algebraic surface is said to be of degree $n = \max(i + j + k)$ where n is the maximum sum of powers of all terms $a_m x^{i_m} y^{j_m} z^{k_m}$
  - Example:
    - Sphere $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - r^2 = 0$
    - Degree $n = 2$

Degree of algebraic curves

- Quadratic curves are algebraic curves of a degree 2
  - Examples: conic sections
- The conic sections are the curves generated by the intersections of a plane with a conical surface
- Types of conic sections:
  - Circle $(x - x_o)^2 + (y - y_o)^2 - r^2 = 0$
  - Ellipse $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} - 1 = 0$
  - Parabola $(y-y_0)^2 - 4a(x-x_0)^2 = 0$
  - Hyperbola

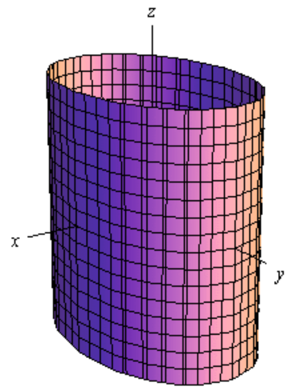
$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} - 1 = 0$$



Quadratic surfaces

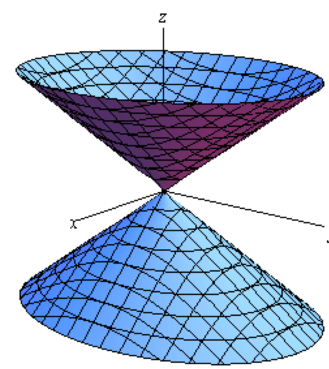
- Quadratic surfaces are algebraic surfaces of a degree 2. Examples:
  - Cylindrical surface

$$\frac{(x-x_0)^2}{r^2}+\frac{(y-y_0)^2}{r^2}-1=0$$



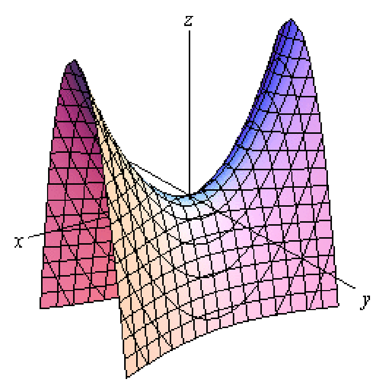
  - Conical surface

$$\frac{(x-x_0)^2}{r^2}+\frac{(y-y_0)^2}{r^2}-(z-z_0)^2=0$$



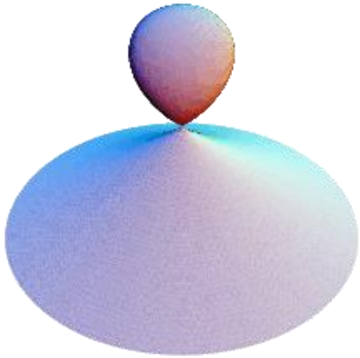
  - Hyperbolic paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z}{c}=0$$



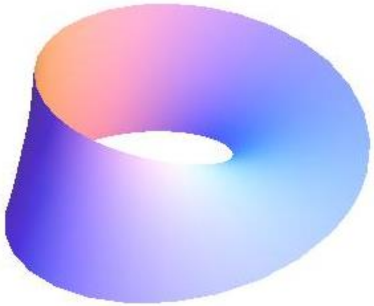
Cubic surfaces

- Cubic surfaces are algebraic surfaces of a degree 3. Examples:
  - Ding-dong surface



$$x^2 + y^2 = (1 - z)z^2$$

  - Möbius strip

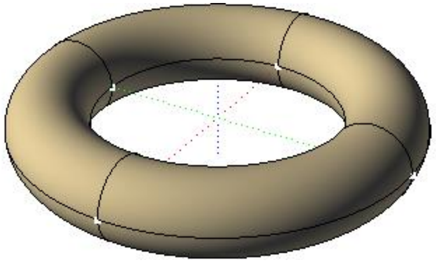


$$x^2 y - R^2 y + y^3 - 2Rxz - 2x^2 z - 2y^2 z + yz^2 = 0$$

Quartic surfaces

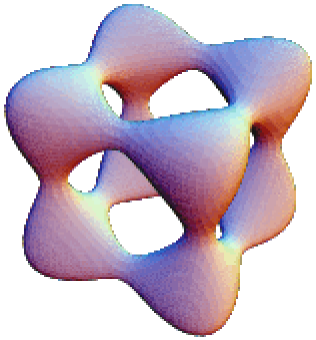
- Quartic surfaces are algebraic surfaces of a degree 4. Examples:
  - Torus: a surface of revolution generated by revolving a circle in three dimensional space about an axis coplanar with the circle. It is assumed that the axis does not touch the circle.

$$\left(R^2+x^2+y^2+z^2-r^2\right)^2-4R^2\left(x^2+y^2\right)=0$$



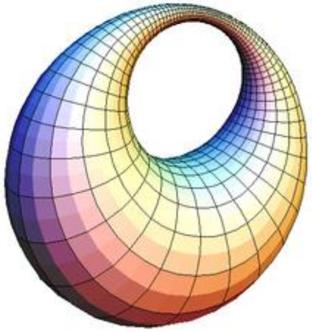
  - Tanglecube

$$x^4-5x^2+y^4-5y^2+z^4-5z^2+11.8=0$$



  - Dupin cyclide

$$\left(x^2+y^2+z^2+b^2-d^2\right)^2-4(ax-cd)^2-4b^2y^2=0$$



Rendering of algebraic curves and surfaces

- Ray-object intersection, general idea:

1. *Parametric* form for the ray
   - $x = x(t)$, $y = y(t)$, $z = z(t)$
2. *Implicit* form for the object
   - $f(x, y, z) = 0$
3. Combine both equations: substitute parametric x, y, z for the line to the implicit equation of the object
4. Solve the resulting equation for *t*
5. Get the resulting points by calculating the points on the line for resulting *t*’s

The equation can be solved if it is quadric, cubic or quartic

Rendering of algebraic curves and surfaces

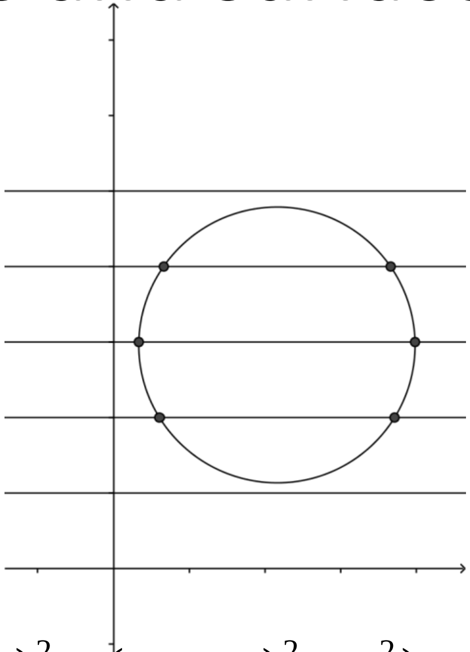
- ***Line-circle intersection***

Line:

$$x = a_x + d_x t$$
$$y = a_y + d_y t$$

Circle:

$$(x - x_0)^2 + (y - y_0)^2 - r^2 = 0$$



$$(a_x + d_x t - x_0)^2 + (a_y + d_y t - y_0)^2 - r^2 = 0$$

$$t^2(d_x^2 + d_y^2) + t(2(a_x - x_0)d_x + 2(a_y - y_0)d_y) + ((a_x - x_0)^2 + (a_y - y_0)^2 - r^2) = 0$$

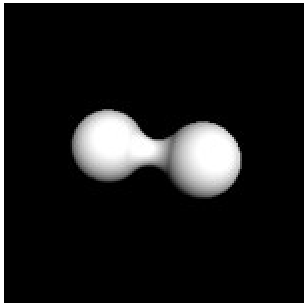
0 roots = no intersection, 1 root – the line touches the circle, 2 roots – the line intersects the circle in two points

Direct rendering for algebraic curve:

- For every line in the viewport window, create the ray starting in the leftmost pixel and ending in the rightmost pixel (or vice versa)
- Intersect with the curve, find point(s), plot these points.

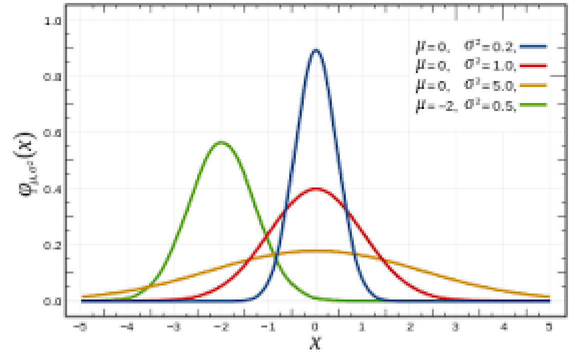
Blobby objects

- By a blobby object we mean a non-rigid object. That is things, like cloth, rubber, liquids, water droplets, etc.
- For example, in a chemical compound electron density clouds tend to be distorted by the presence of other atoms/molecules.
- Several models have been developed to handle these kind of objects.



Blobby objects

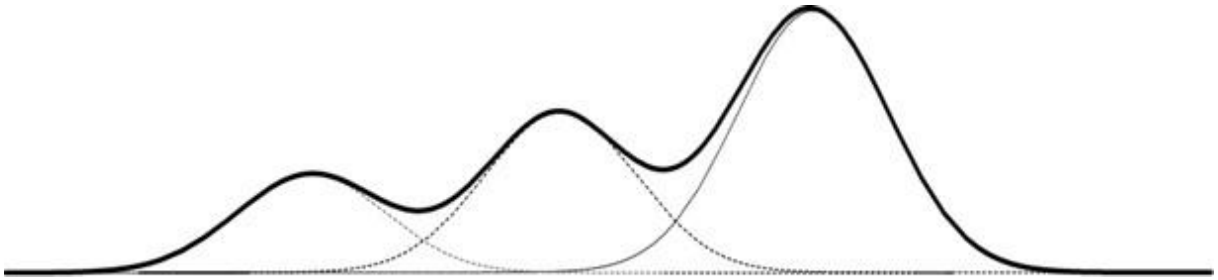
- One technique is to use a combination of Gaussian density functions (Gaussian bumps)



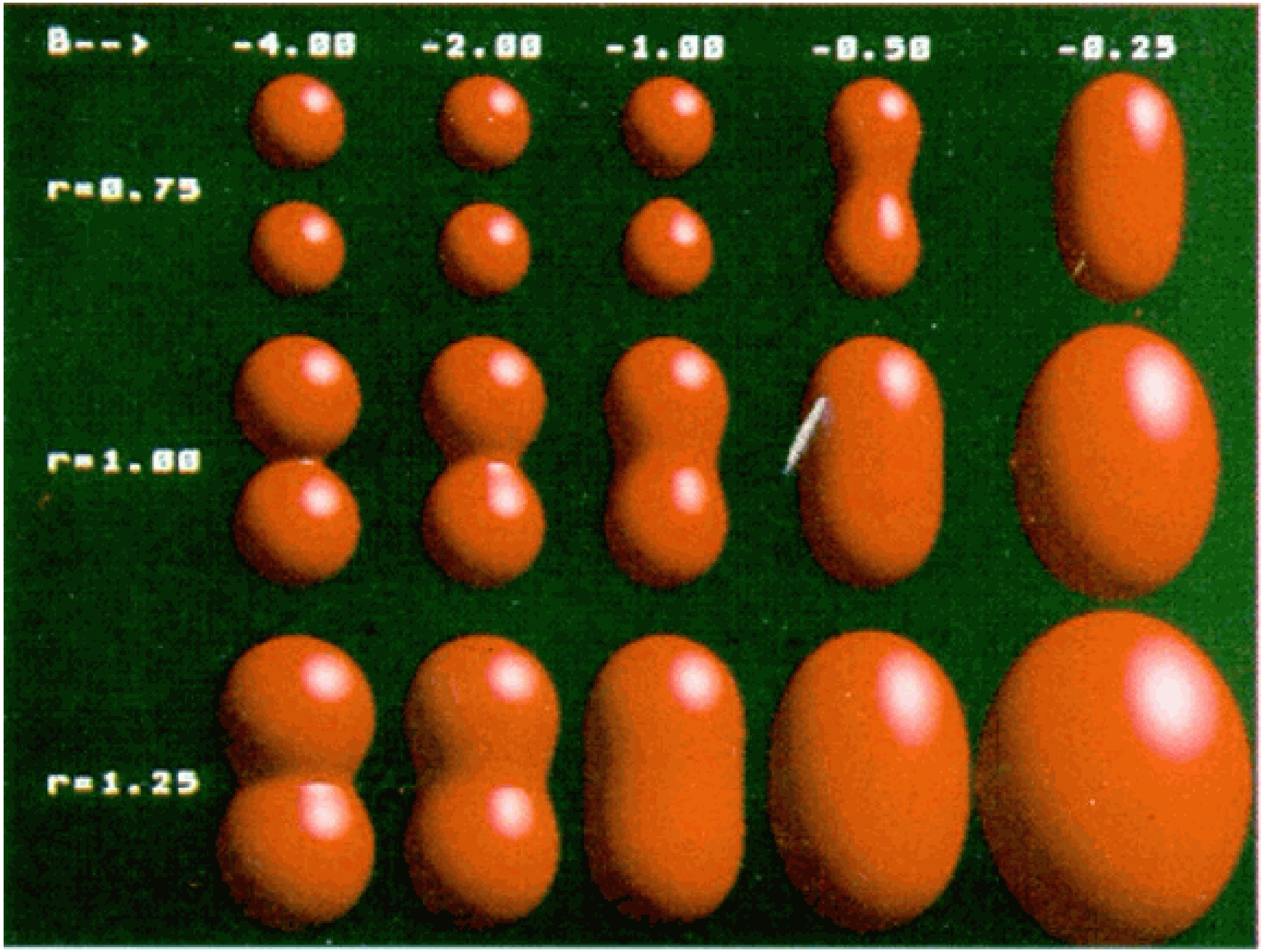
$$f(x, y, z) = \sum_k b_k e^{-a_k r_k^2} - T = 0$$

where $r_k^2 = x_k^2 + y_k^2 + z_k^2$

and T is a threshold



Metaballs (Blinn Blobbies)



Ray-traced metaballs

- Image courtesy of Ange Gabriel, rendered on Bryce Render+V4