Geometric transformations on the plane

Aim: Analysis of performing the geometric transformations on the plane.

Task: Implement software for animation of shape on the plane using matrix representation of the geometric transformations.

Result: The executed binary file. The source code. The report.

Theoretical part:

Consider the system of equations:

$$\begin{cases} x' = x + a; \\ y' = y. \end{cases}$$

We can interpret these equations in ambiguous manner:

1. All points of plane *xy* shift by a translation to the right on distance *a* — Fig. 2.1(a);
2. Coordinate axis *x* and *y* shift to the left on distance *a* — Fig. 2.1(b).

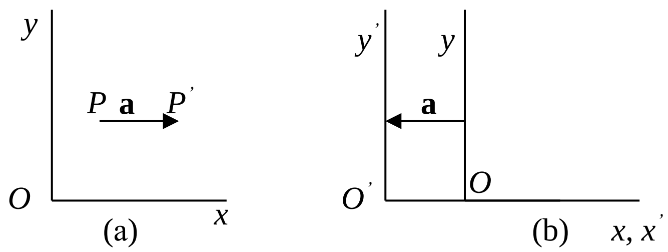

Fig. 2.1. (a) — Translation; (b) — Changing of coordinates

This example shows the basic approach, which we will use with more complicated situations, to consider the system of equations as the transformations of all points in fixed coordinate system, or as changing of the coordinate system.

Consider the rotation of the point P (x,y) around origin O on the angle φ. The new point will be denoted as P' (x', y') as in Fig.2.2.

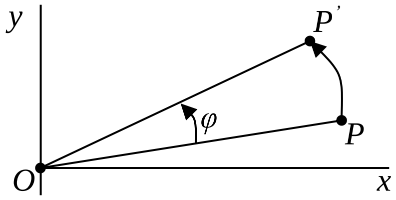

Fig. 2.2. Rotation around point O to an angle φ

Equations for the rotation of points on the plane about the origin O:

$$\begin{cases} x' = x\cos\varphi - y\sin\varphi; \\ y' = x\sin\varphi + y\cos\varphi \end{cases}$$

For the rotation about any point (x_0, y_0), x should be changed to $(x - x_0)$, y -> $(y - y_0)$, x' -> $(x' - x_0)$, y' –> $(y' - y_0)$:

$$\begin{cases} x' - x_0 = (x - x_0)\cos\varphi - (y - y_0)\sin\varphi; \\ y' - y_0 = (x - x_0)\sin\varphi + (y - y_0)\cos\varphi \end{cases}$$

$$\begin{cases} x' = x_0 + (x - x_0)\cos\varphi - (y - y_0)\sin\varphi; \\ y' = y_0 + (x - x_0)\sin\varphi + (y - y_0)\cos\varphi \end{cases}$$

The system of equations:

$$\begin{cases} x' = ax; \\ y' = by \end{cases}$$

describes changing the scale relatively the origin O.
If $a = b$ then we can say about uniformly scaling without distortion of the image. Otherwise, transformation is called non-uniform scaling.

For scaling about the point (x_0, y_0), x should be changed to $x - x_0$, y –> $y - y_0$, x' -> $x' - x_0$, y' –> $y' - y_0$:

$$\begin{cases} x' - x_0 = a(x - x_0); \\ y' - y_0 = b(y - y_0 \end{cases}$$

$$\begin{cases} x' = x_0 + a(x - x_0); \\ y' = y_0 + b(y - y_0) \end{cases}$$

The system of equations of the rotation of point about origin can be wrote as a matrix equation:

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}$$

The matrix equation for scaling:

$$[x' \quad y'] = [x \quad y]\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

2D homogeneous coordinates.

2D homogeneous coordinates of the point have following view:

$$[X \quad Y \quad W]$$

where W is a coefficient is not equal 0.

2D Cartesian coordinates of point *(x, y)*, we can get from 2D homogeneous coordinates by dividing by coefficient W:

$$x = X/W;\ y = Y/W;\ W \neq 0$$

Homogeneous coordinates can be represented as scaled Cartesian coordinates with coefficient W located on the plane Z = W.

There isn't a single representation of a point in Cartesian coordinates, because we can take any value for W.

Translation. Let's P (*x*, *y*) move to P' (*x'*, *y'*),

$$x' = x + a;$$

$$y' = y + b.$$

This equations can be write in matrix view:

$$[x' \quad y'] = [x \quad y \quad 1]\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a & b \end{bmatrix}$$

Or:

$$[x' \quad y' \quad 1] = [x \quad y \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{bmatrix}$$

This equations are equivalent.

This expression is expression in "homogeneous coordinates".

Homogeneous coordinates allow all transformations (translation, scaling and rotation) to be expressed homogeneously, allowing composition via multiplication by 3x3 matrices.

Let's combine several operations (rotation and 2 translations).

The equation for the rotation around point O to an angle φ:

$$[x' \quad y' \quad 1] = [x \quad y \quad 1]\begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The equation for the rotation around point (x_0, y_0) to an angle φ:

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix}\mathbf{R}$$

,where **R** is matrix 3x3.

For finding matrix **R**, let's partition the action in three steps with intermediate points (u_1, v_1) and (u_2, v_2).

1. Expression for the translation origin O to point (x_0, y_0):

$$\begin{bmatrix} u_1 & v_1 & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix}\mathbf{T'}$$

$$\mathbf{T'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_0 & -y_0 & 1 \end{bmatrix}$$

2. Rotation around point O to an angle φ:

$$\begin{bmatrix} u_2 & v_2 & 1 \end{bmatrix} = \begin{bmatrix} u_1 & v_1 & 1 \end{bmatrix}\mathbf{R}_0$$

$$\mathbf{R}_0 = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Return origin O back:

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} u_2 & v_2 & 1 \end{bmatrix}\mathbf{T}$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_0 & y_0 & 1 \end{bmatrix}$$

Possibility of composition these steps is based on the associative properties of matrix multiplication, (AB)C = A (BC) for any A, B, C, with corresponding sizes. For any part of these equation we can write ABC. Let's find:

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix}\mathbf{R}$$

where

$$\mathbf{R} = \mathbf{T'}\mathbf{R}_0\mathbf{T}$$

It will be desired matrix:

$$\mathbf{R} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ c_1 & c_2 & 1 \end{bmatrix}$$

where:

$$c_1 = x_0 - x_0\cos\varphi + y_0\sin\varphi$$

$$c_2 = y_0 - x_0\sin\varphi - y_0\cos\varphi$$

Thus, we can write any sequence of the geometric operations on the plane.

Tasks:

You need to use of an array of vectors for the specifying of the shape for the transformation. Every vector contains the coordinates of the one vertices of shape. A necessary transformation should be specified by matrix 3x3. A result transformation matrix should be specified by composition of multiplication of corresponded matrices.

It is necessary to implement the task using a general approach with parameters (rotation angle, value of translation, scale coefficient) for an arbitrary shape, and implement the transformation matrix for your version of the task.

Bordered lines in figures show the first state of the shape.

Use timer for moving your shape.

Exercise 1 (pendulum):

Construct the shape using straight lines (bordered lines in Figure 1). First position – shape tilted to an 45 degree from vertical.

Rotate the shape around the point (axis of pendulum) with angle step 5 degrees in a counter-clockwise direction, from far-left position to far-right position. After that, rotate the shape in the reverse direction to far-left position. Repeat the process of the oscillation of pendulum.

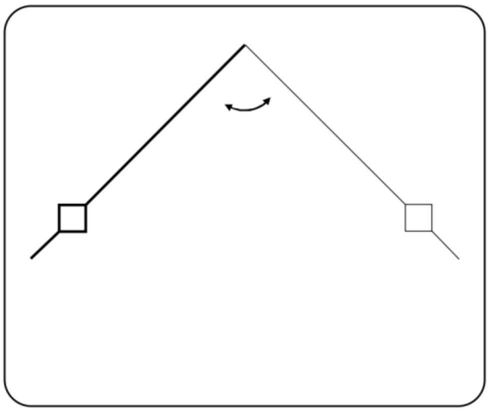

Figure 1. The pendulum

Exercise 2 (Ferris wheel):

Construct the shape using triangle and rectangle (bordered lines in Figure 2).

Rotate the shape around of the screen center with angle step 15 degrees. Implement the several complete rotations of shape.

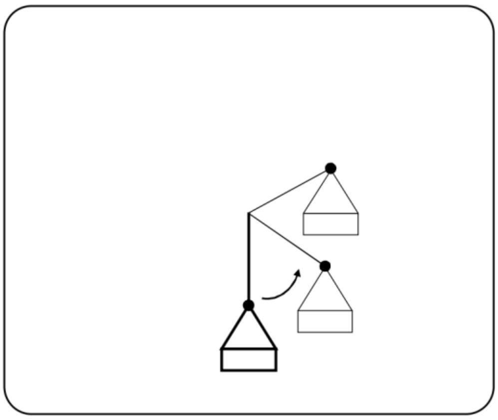

Figure 2. Ferris wheel

Exercise 3 (wheel):

Construct the shape using straight lines as bicycle wheel (bordered lines in Figure 3). Rotate the shape around axis of wheel with angle step 22.5 degrees in a clockwise direction, and move shape from far-left position to far-right position.

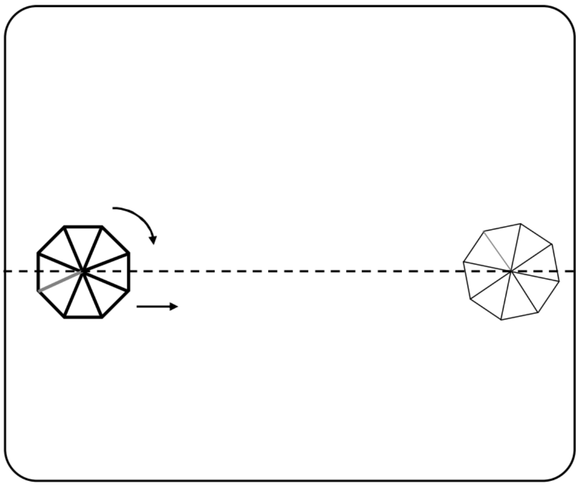

Figure 3. The wheel

Exercise 4 (hands of the clock):

Construct the shape using straight lines as hands of the clock (bordered lines in Figure 4). Rotate the shape around axis with angle step 6 degrees in a clockwise direction.

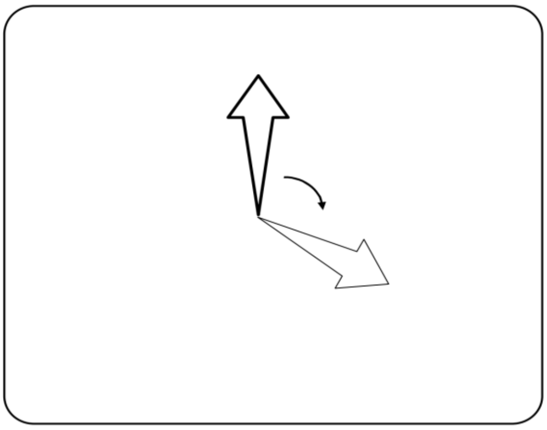

Figure 4. Hands of the clock

Exercise 5 (Leaf falling):

Construct the shape using straight lines like a tree leaf (bordered lines in Figure 5). Rotate the shape around point P (upper center point of screen) with angle step 10 degrees from far-left position to far-right position and vice versa. Perform the rotation and scaling of the shape.

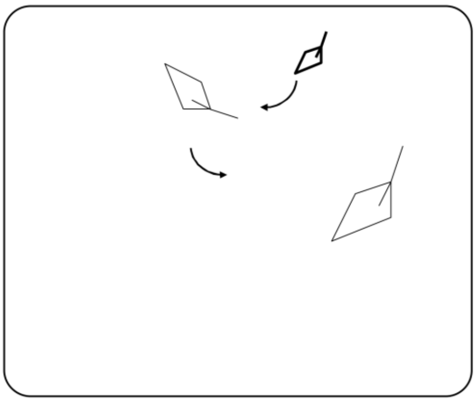

Figure 5. Leaf falling

Exercise 6 (Rectangle flip):

Construct the rectangle using straight lines (bordered lines in Figure 6). Rotate the shape around point P (lower right corner of the new rectangle) with angle step 90 degrees.

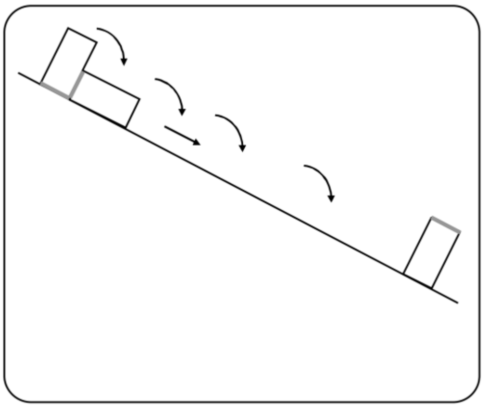

Figure 6. Rectangle flip