Construction of perspective projection of 3D scene

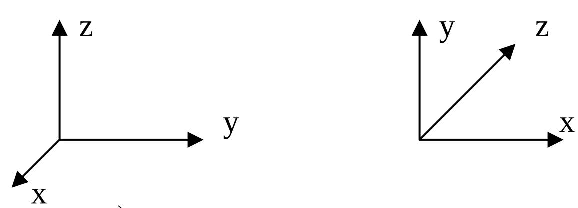
<u>Aim:</u> The study of the process of the construction of parametric skeleton model of the 3D object, and creation of parallel and perspective projections.

<u>Task:</u> Implement software for the construction and visualization of the parametric skeleton model of the 3D object using parallel and perspective projections.

<u>Result:</u> The executed binary file. The source code. The report.

Theoretical part:

We will use two coordinate systems in our software applications. The right coordinate systems of vector algebra will be used for specifying the point coordinates at 3D space of model of the world (Fig. 3.1a).



a)

Fig. 3.1. Right (a) and left (b) coordinate system.

Left coordinate system will be used for representation of the point coordinates in the coordinate system which is connected with the screen plane (Fig. 3.1b). The plane of screen is superposed with the XY plane, and the direction of Z axis is in screen depth. The rotations will be positive in left coordinate system, if to look from positive semi-axis.

3D geometric transformations

3D geometric transformations work similar to the transformation on the plane, and an arbitrary transformation of point coordinates or changing of coordinate systems can be represent as a sequence of elementary transformations: translation, rotation, scaling.
Matrices for these transformations in homogeneous coordinate system are the following:

Translation of point: $\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{bmatrix}$

Translation of coordinate system origin: $\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a & -b & -c & 1 \end{bmatrix}$

Rotation of point around OZ axis to an angle φ: $\mathbf{R_z} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Rotation of coordinate system around OZ axis to an angle φ: $\mathbf{R_z}^{-1} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Rotation of point around OX axis to an angle φ: $\mathbf{R_x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Rotation of coordinate system around OX axis to an angle φ: $\mathbf{R_x}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Rotation of point around OY axis to an angle φ: $\mathbf{R_y} = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Rotation of coordinate system around OY axis to an angle φ: $\mathbf{R_y}^{-1} =$ $$\begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling of points: $\mathbf{S} = \begin{bmatrix} Sx & 0 & 0 & 0 \\ 0 & Sy & 0 & 0 \\ 0 & 0 & Sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Scaling of coordinate system: $\mathbf{S}^{-1} = \begin{bmatrix} \frac{1}{Sx} & 0 & 0 & 0 \\ 0 & \frac{1}{Sy} & 0 & 0 \\ 0 & 0 & \frac{1}{Sz} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

View transformation. View Reference Coordinates.

In this work, we use skeleton representations of the scene objects. Every object of scene is specified by a set of vertices and edges. The vertices are specified by 3D coordinates in the world coordinate system. Such coordinates are called the world coordinates. Every edge of the object will be specified by two vertices and will represent a segment of straight line (Fig. 3.2).

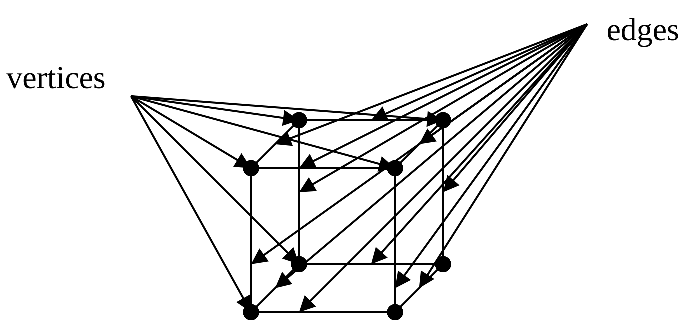


Fig. 3.2. Vertices and edges of the skeleton model of an object.

There are following conditions for getting of an object view from an observation point:

1. Observation point coordinates in the world coordinate system.
2. The vector of the observation direction.
3. The screen plane which is used for the creation of representation.

In our case, we use following conditions (Fig 3.3):
1. The origin of the world coordinate system is a center of the cube.

2. The direction of observation vector is from observation point E to origin of the world coordinate system.
3. The screen plane perpendicular of observation vector, and this vector intersects it in the screen center.

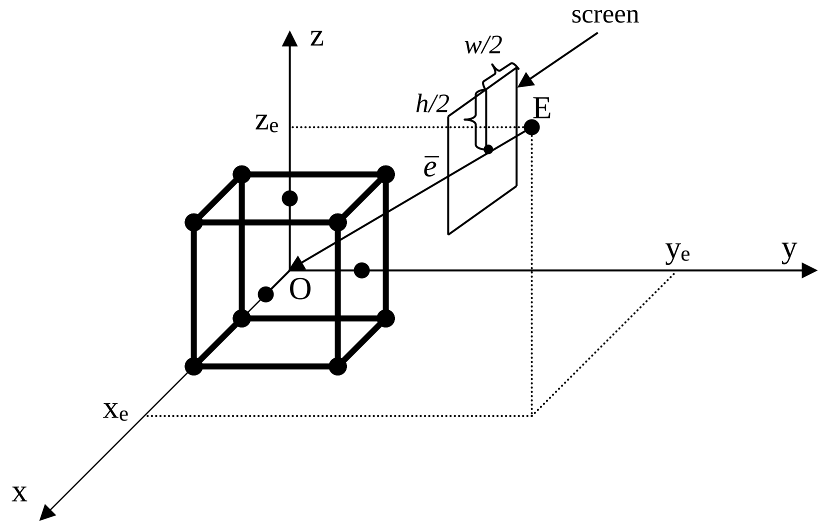


Fig 3.3. The objects of the scene.

The following actions are required for getting of the cube view:

1. Rotate the world coordinate system around of Z and Y axes in such manner that direction of X axis will be to the point E.
2. Translate the origin of this coordinate system to the point E.
3. The reverse of the direction of X axis.
4. Rename of X axis to Z, Y to X, Z to Y.
5. Locate the screen on the distance d from the observation point of new Z axis in such way that X axis of the screen will be parallel to X axis of new coordinate system, and Y axis of the screen will be parallel to Y axis of new coordinate system.

The sequence of these transformations is called a view transformation**,** and coordinate system is called a view reference coordinates system (VRC).

Now, we can get the view of the parallel projection of cube (representation of cube from observation point E) and draw the scene using view reference coordinates X and Y and ignore Z. For this we need to recalculate of the coordinates of the object vertices using the view reference coordinates system**.**

View transformation is specified by Matrix 4x4, which is produced by multiplication of following four matrices of the elementary transformations:

Rotation of point around OZ axis to an angle φ

1. Matrix R^{-1}_z - Rotation matrix of coordinate system around Z axis to the angle θ (Fig 4).

2. Matrix $R^{-1}{}_y$ - Rotation matrix of coordinate system around Y axis to the angle $\frac{\pi}{2} - \varphi$.
3. Matrix $T^{-1}{}_e$ - translation of the origin of coordinate system to the point E.
4. Matrix S - Transformations of coordinate system axis.

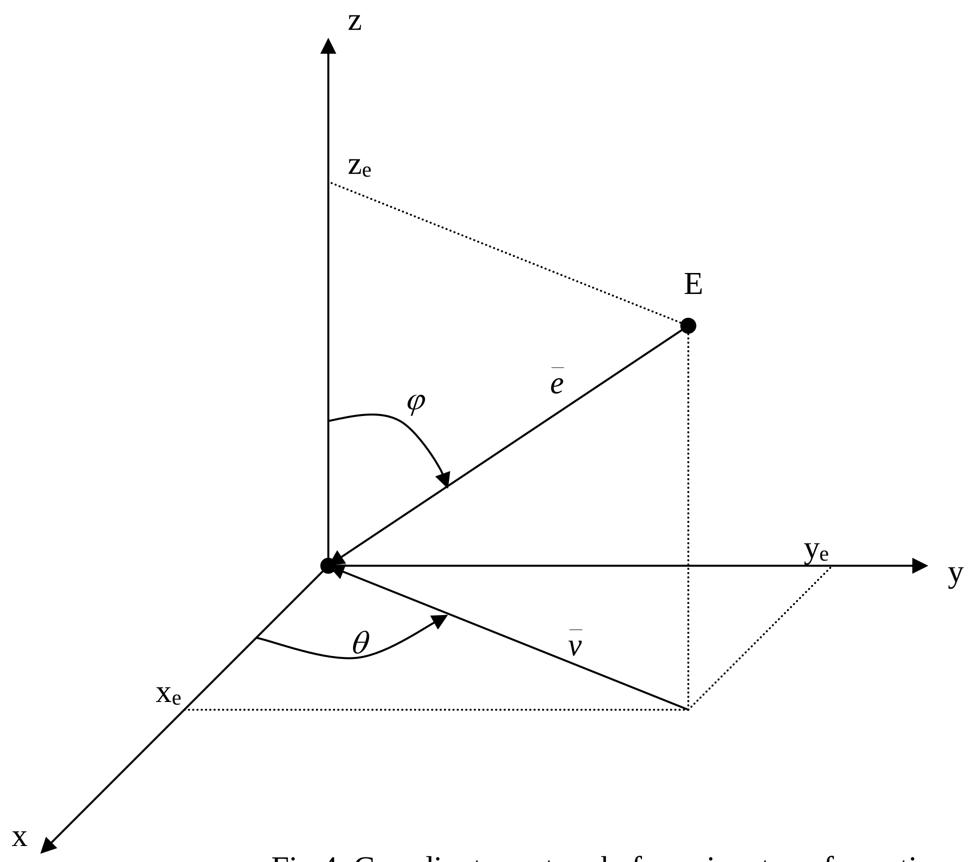


Fig.4. Coordinate system before view transformation.

Matrix of rotation of coordinate system around Z axis to the angle θ is following:

$$R^{-1}{}_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\cos\theta = \frac{x_e}{|\bar{v}|}$, a $\sin\theta = \frac{y_e}{|\bar{v}|}$ (See Fig. 4)

where $|\bar{v}| = \sqrt{x_e^2 + y_e^2}$

The coordinate axis after rotation (Fig. 5).

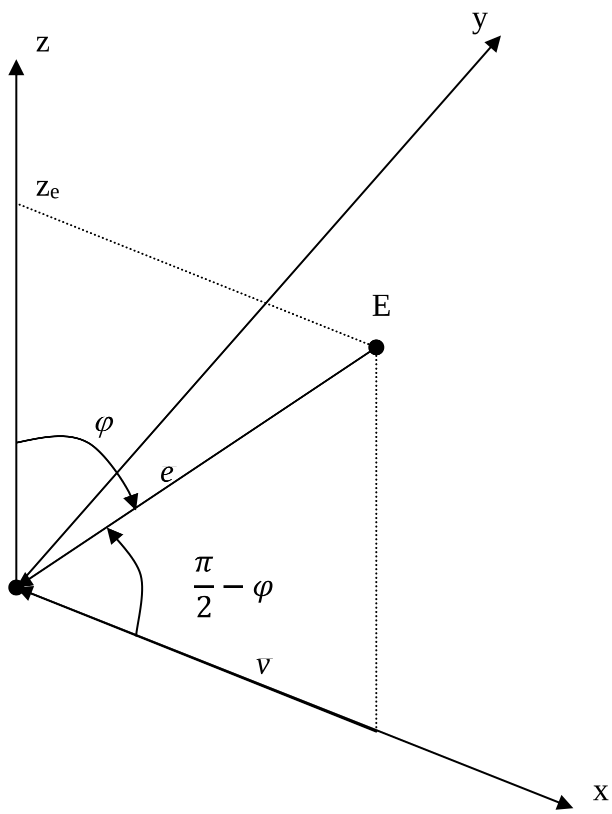


Fig 5. Coordinate system after rotation around Z axis.

Now, let's rotate coordinate system to the angle $\frac{\pi}{2} - \varphi$ around Y axis. Matrix for this transformation is following:

$$R^{-1}_y = \begin{bmatrix} \cos\frac{\pi}{2} - \varphi & 0 & \sin\frac{\pi}{2} - \varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\frac{\pi}{2} - \varphi & 0 & \cos\frac{\pi}{2} - \varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\cos\varphi = \frac{z_e}{|\bar{e}|}$, a $\sin\varphi = \frac{|\bar{v}|}{|\bar{e}|}$ (see Fig.5).

Where $|\bar{v}| = \sqrt{x_e^2 + y_e^2}$, a $|\bar{e}| = \sqrt{x_e^2 + y_e^2 + z_e^2}$.

The coordinate system after this transformation (Fig.6).

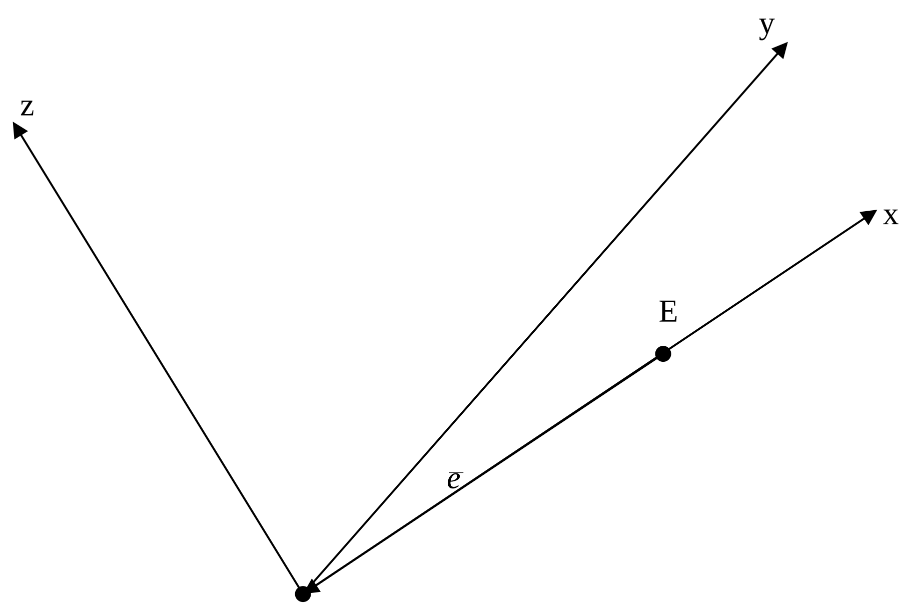


Fig.6. Coordinate system after rotation around Y axis.

Let's translate the origin of coordinate system to the point E. Matrix for this transformation is the following:

$$T^{-1}_{e} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -|\bar{e}| & 0 & 0 & 1 \end{bmatrix}, \text{ where } |\bar{e}| = \sqrt{x_e^2 + y_e^2 + z_e^2}\ .$$

The position of coordinate axis after translation the origin of coordinate system to the point E (Fig. 7).

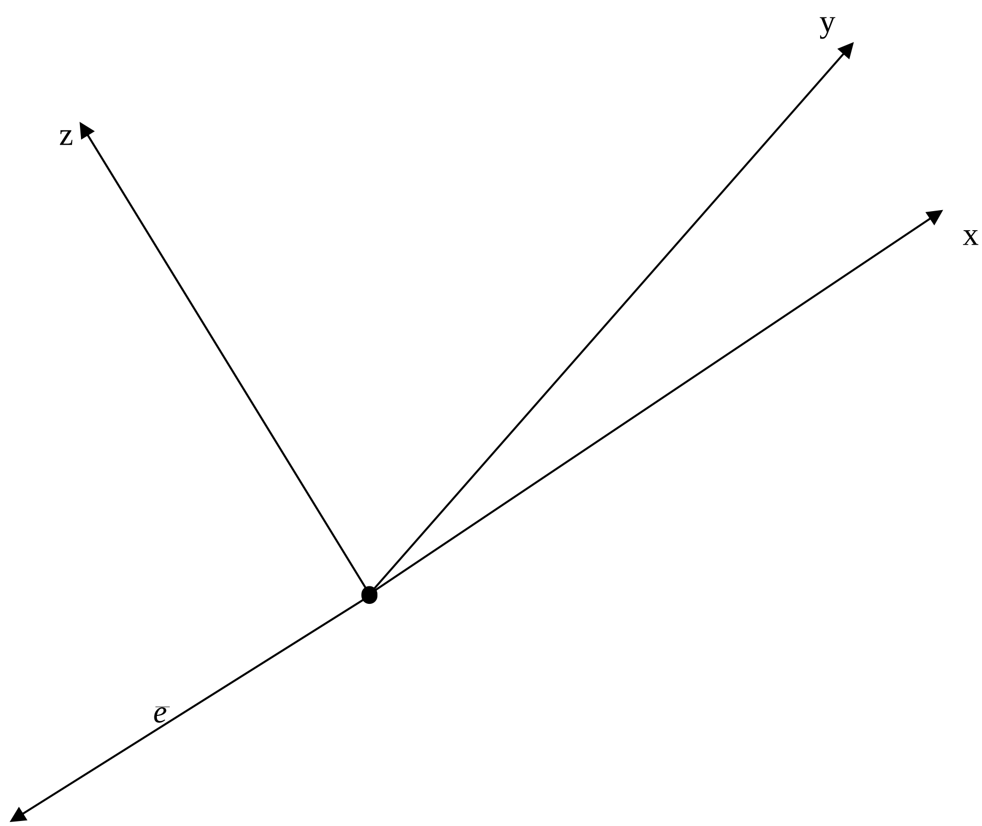


Fig. 7. The position of coordinate axis after translation the origin of coordinate system to the point E.

Now, let's change the direction of X axis and rename coordinate axis. Matrix of changing direction of X axis:

$$S = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The result of all transformation is shown in Fig.8.

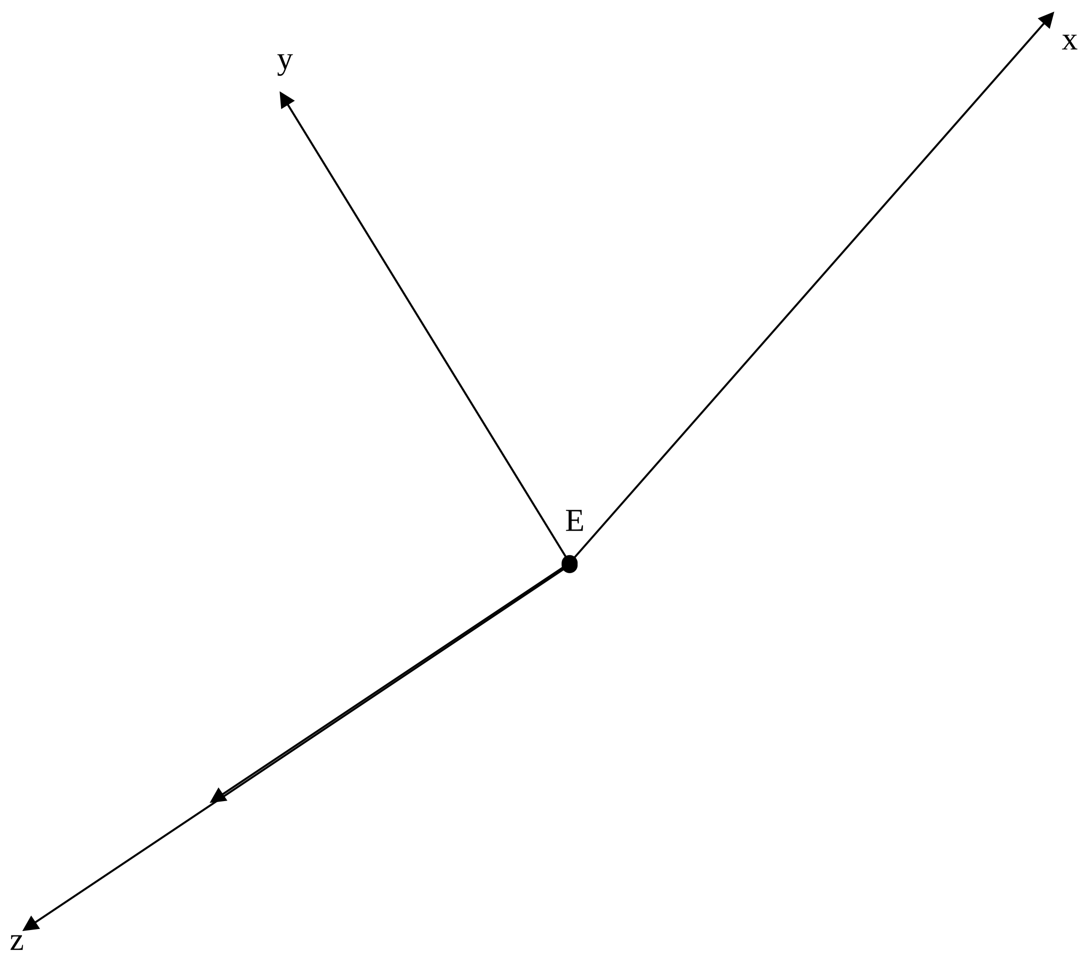


Fig. 8. The position of axis of the view reference coordinates system after all transformations.

Homography (projective transformation)

Let’s apply the result matrix to object coordinates, after that we can get coordinates of object vertices in the view reference coordinates system.
For getting the perspective projection of object, we need to calculate new coordinates of X and Y object vertices with taking into account the distance to observation point (Fig. 9).

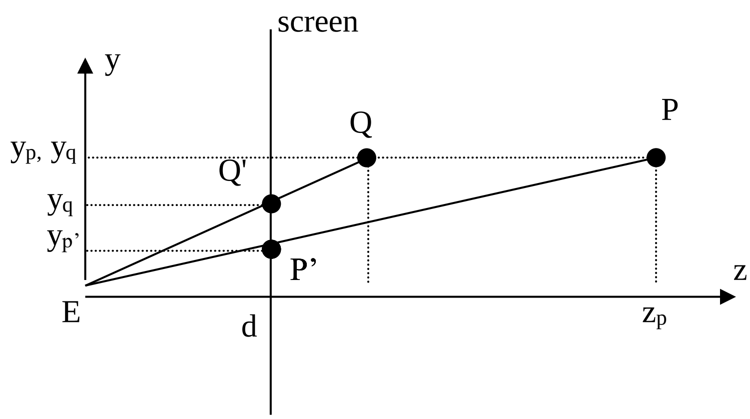


Fig. 9. The calculation of the perspective projection.

Consider the triangles EPZ_p and EP'd. This triangles are similar to each other, as result: $\frac{d}{z_p} = \frac{y_{p'}}{y_p}$,

=> $y_{p'} = \frac{dy_p}{z_p}$.

Thus, if point P is specified by view reference coordinates X,Y,Z, then coordinates x' and y' of the central projection of this point to the screen plane are following:

$$\begin{cases} x' = \dfrac{d \bullet x}{z} \\ y_{p'} = \dfrac{d \bullet y}{z} \end{cases}$$

Where d – distance from observation point to screen.

Last step is the recalculation of getting coordinates of projections of points in screen coordinates. The screen coordinates Sx and Sy are calculated as follows:

$$\begin{cases} Sx = x' + \dfrac{W}{2} \\ Sy = y' + \dfrac{H}{2} \end{cases}$$

where W is the width of the screen, H is the height of the screen.

Tasks:

You need to get parallel and perspective projections of 3D scene. The representation object is the skeleton 3D model, which consists of vertices coordinates and segments of straight lines (edges), which connect these vertices (Fig.10).

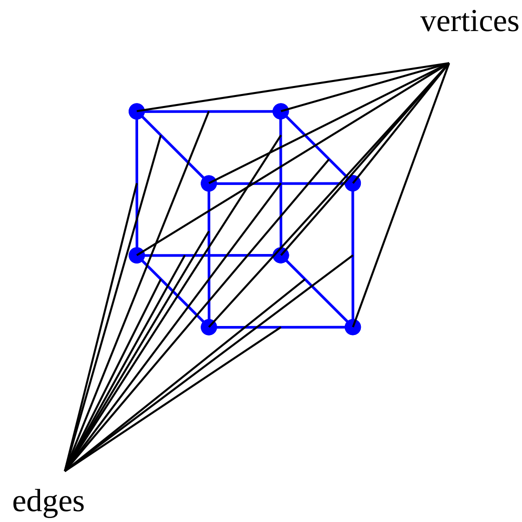


Fig.10. The skeleton model of the cube.

Your model of the object is set by your version of the task. The object is specified using parametric method, i.e., you need to calculate the vertex coordinates of the object using the world coordinate systems and construct the list of edges.

An example of the object is a cube (Fig.11) with the side *a*. The center of symmetry of the cube is in origin point *O*. Edges of the cube are parallel to corresponding axis of the coordinate system.

For creation of the 3D projection of the scene, require following information: an observation point coordinates in the world coordinate system; an observation direction; location of screen coordinates of the computer relatively to the world coordinate system (Fig. 12). Coordinates of observation point are specified by the user.

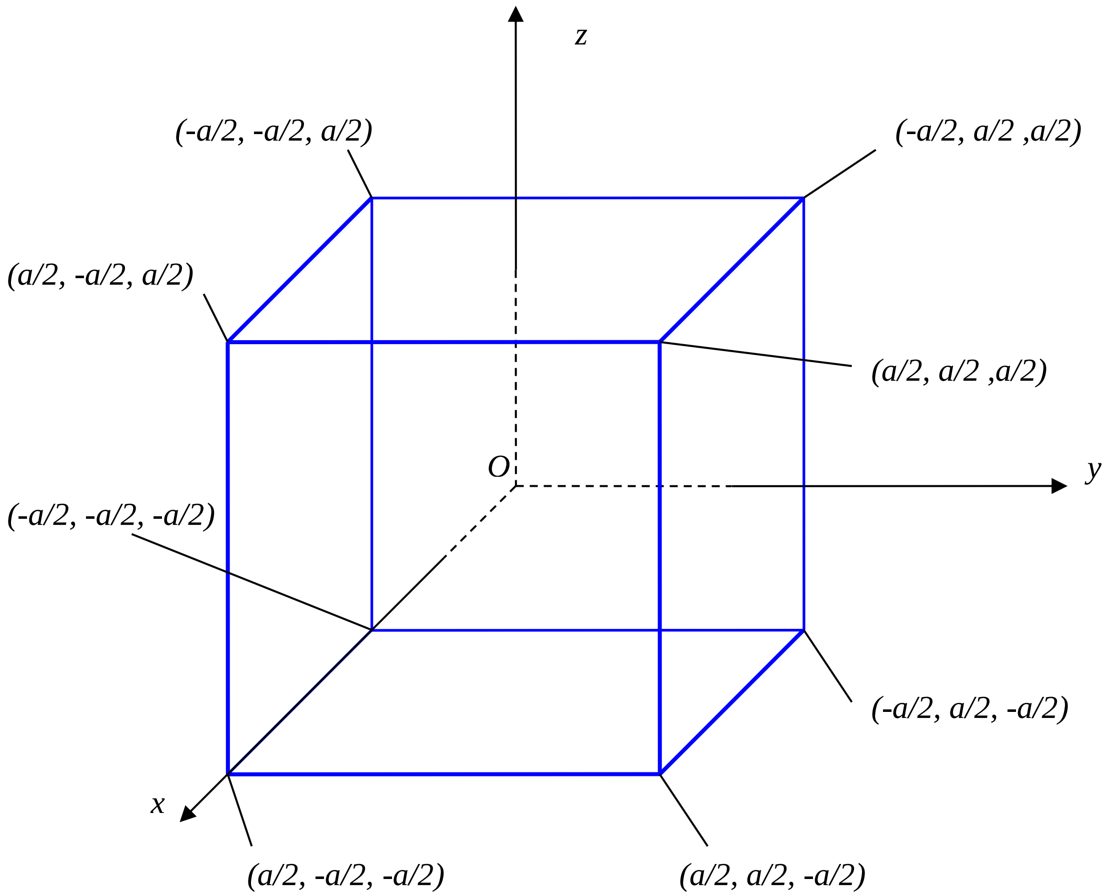


Fig.11. Parametrically specified cube.

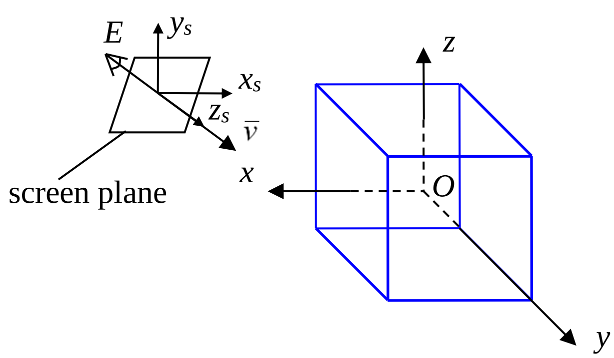
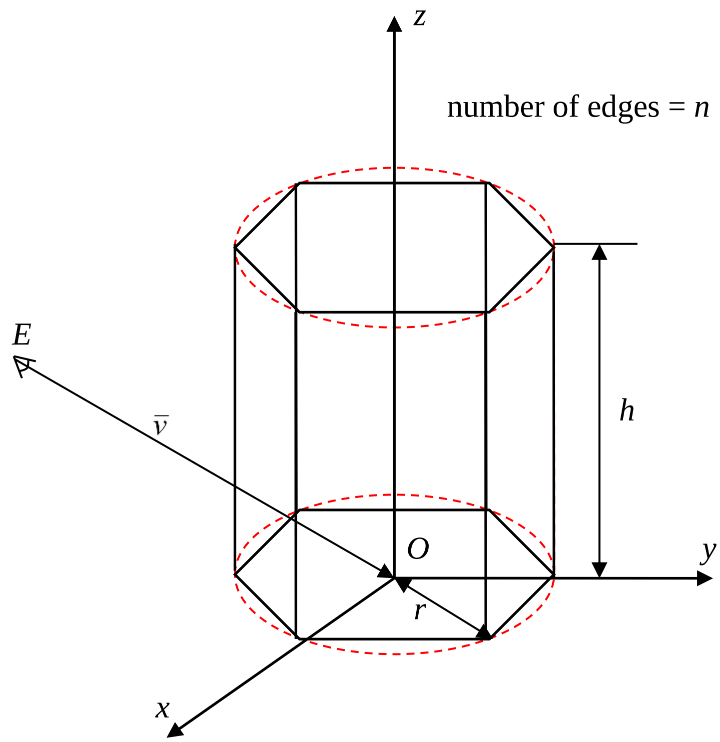


Fig.12. Projection of 3D scene.

Exercise 1:

The object is a cylinder approximated by a correct n-edge prism. The both bases of the cylinder are parallel of the XY plane. The bottom base lies on the plane XY. Center of bottom base matches with origin point *O* of the world coordinate system. Number of edges of the prism – *n*, radius of the circumference around the base - *r*, the height of object – *h*, coordinates of observation point E are specified by the user. The observation vector $\bar{v}$ is directed to point O.

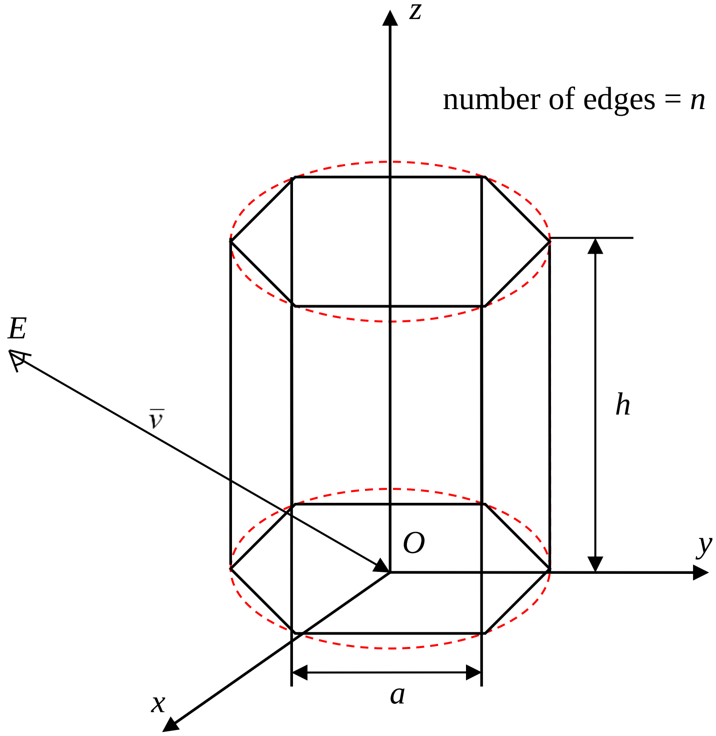
Construct parallel and perspective projections for this scene.



Exercise 2:

The object is a cylinder approximated by a correct n-edge prism. The both bases of a cylinder are parallel of the XY plane. The bottom base lies on the plane XY. The center of the bottom base matches with the origin point *O* of the world coordinate system. Number of edges of the prism – *n*, length of the base side - *a*, the height of object – *h*, coordinates of observation point E are specified by user. The observation vector $\bar{v}$ is directed to point O.

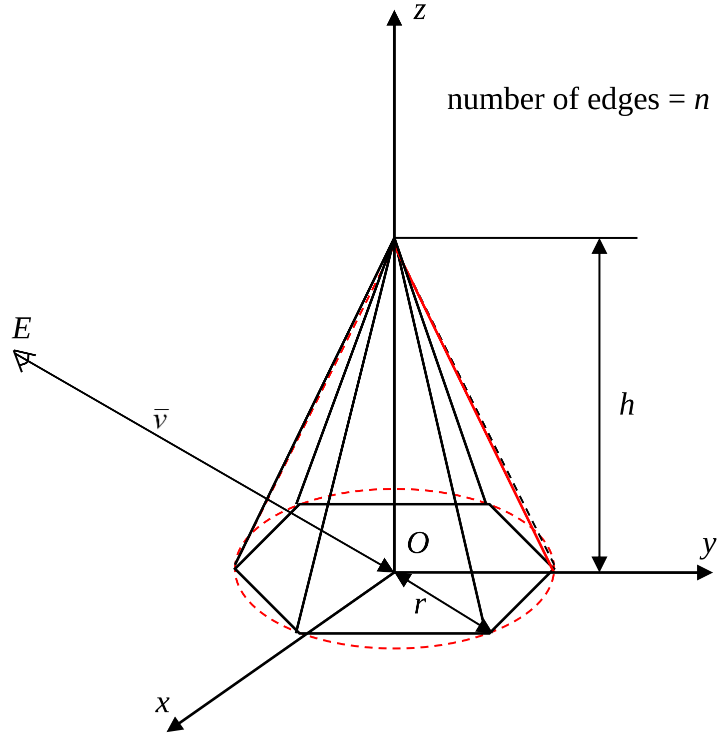
Construct parallel and perspective projections for this scene.



Exercise 3:

The object is a cone approximated by a right n-edge pyramid. The base of a cone is parallel to the XY plane. The center of the base matches with the origin point of the world coordinate system. Number of edges – *n*, radius of the circumference around the base - *r*, the height of object – *h*, coordinates of observation point E are specified by the user. The observation vector $\bar{v}$ is directed to point O.

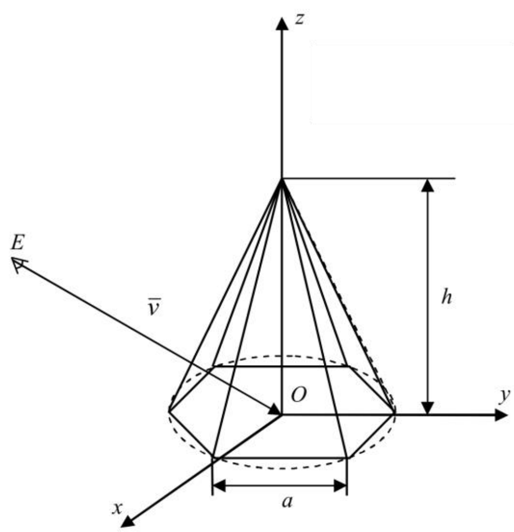
Construct parallel and perspective projections for this scene.



Exercise 4:

The object is a cone approximated by a right n-edge pyramid. The base of a cone is parallel to the XY plane. The center of the base matches with the origin point of the world coordinate system. Number of edges – *n*, length of the base side - *a* , the height of object – *h*, coordinates of observation point E are specified by the user. The observation vector $\bar{v}$ is directed to point O.

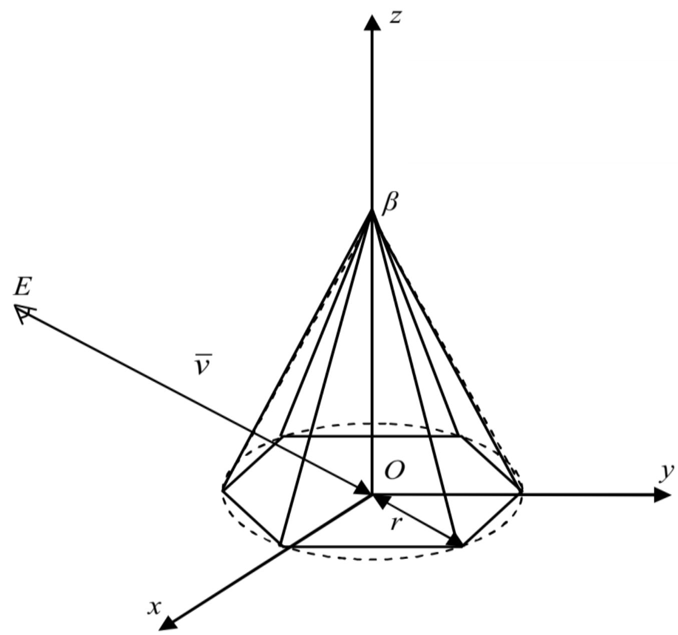
Construct parallel and perspective projections for this scene.



Exercise 5:

The object is a cone approximated by a right n-edge pyramid. The base of a cone is parallel to the XY plane. The center of the base matches with the origin point of the world coordinate system. Number of edges – *n*, radius of the circumference around the base - *r* , cone opening angle - β, the height of object – *h,* coordinates of observation point E are specified by the user. The observation vector $\bar{v}$ is directed to point O.

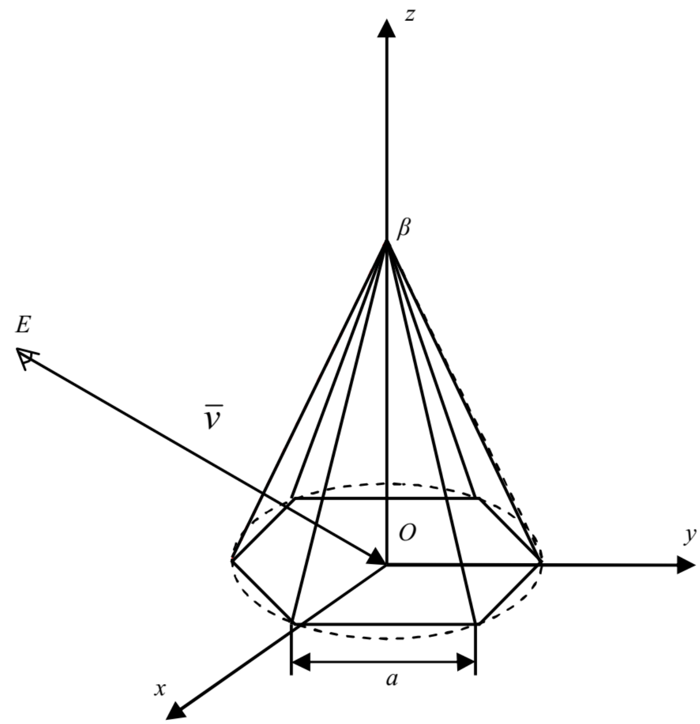
Construct parallel and perspective projections for this scene.



Exercise 6:

The object is a cone approximated by a right n-edge pyramid. The base of a cone is parallel to the XY plane. The center of the base matches with the origin point of the world coordinate system. Number of edges – *n*, length of the base side - *a,* cone opening angle - β, the height of object – *h,* coordinates of observation point E are specified by the user. The observation vector $\bar{v}$ is directed to point O.

Construct parallel and perspective projections for this scene.



z
β
E
$\bar{v}$
O
y
x
a